Math 2203 - Tutorial #10

1. [#4 From Tutorial #9]

2. Find the general solution of $X' = \underbrace{\begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}}_{A} X.$

Eigenvalues:

$$0 = \begin{vmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix} = 5-\lambda \begin{vmatrix} -\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} - \begin{vmatrix} -4 & 0 \\ 2 & 5-\lambda \end{vmatrix}$$

$$= (5-\lambda)[-5\lambda+\lambda^2-4] - [-20+4\lambda] = (5-\lambda)[\lambda^2-5\lambda-4]+4(5-\lambda)$$

$$= (5-\lambda)[\lambda^2-5\lambda] = \lambda(5-\lambda)^2 \Rightarrow \lambda=0,\ \underbrace{\lambda=5}_{\text{double root}}$$

Eigenvectors:

$\lambda=0$: $\left[\begin{array}{ccc:c} 5 & -4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right] r_1 \leftarrow r_1 - 5r_2$

$\left[\begin{array}{ccc:c} 0 & -4 & -10 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right] r_1 \leftarrow r_1 + 2r_3$ $\left[\begin{array}{ccc:c} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right]$ $\left.\begin{array}{l} x=-2z \\ 2y=-5z \\ z=t \end{array}\right\}$ $\begin{array}{l} x=-2t \\ y=-\frac{5}{2}t \\ z=t \end{array}$

$\begin{bmatrix} -2 \\ -5/2 \\ 1 \end{bmatrix} t$ eigenvector. I'll choose $t=2$: $\begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$.

$\lambda=5$: $\left[\begin{array}{ccc:c} 0 & -4 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right]$ $\begin{cases} 2y=0 \Rightarrow y=0 \\ x-5y+2z=0 \end{cases}$

$\Rightarrow x + 2z = 0$

$\begin{cases} x=-2z=-2t \\ z=t \end{cases}$

$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t.$

We get 2 linearly independent solutions

$$e^{0t}\begin{bmatrix}-4\\-5\\2\end{bmatrix} = \begin{bmatrix}-4\\-5\\2\end{bmatrix} \quad \& \quad e^{5t}\begin{bmatrix}-2\\0\\1\end{bmatrix}, \text{ but since}$$

$\lambda=5$ was a root of multiplicity 2 & we only could find one eigenvector corresponding to $\lambda=5$, we need to find one more solution.

For $\lambda=5$, we consider $e^{5t}\begin{bmatrix}-2\\0\\1\end{bmatrix}$, & find another solution of the form $e^{5t}\left(\begin{bmatrix}-2\\0\\1\end{bmatrix}t + P_1\right)$

for some P_1. If this is a solution, it implies that $(A-5I)P_1 = \begin{bmatrix}-2\\0\\1\end{bmatrix}$.

$$\left[\begin{array}{ccc|c}0 & -4 & 0 & -2\\ 1 & -5 & 2 & 0\\ 0 & 2 & 0 & 1\end{array}\right] \xrightarrow{r_1 \leftarrow r_1 + 2r_3} \left[\begin{array}{ccc|c}0 & 0 & 0 & 0\\ 1 & -5 & 2 & 0\\ 0 & 2 & 0 & 1\end{array}\right]$$

$$\begin{aligned}2y &= 1\\ x-5y+2z &= 0\end{aligned} \Rightarrow \begin{aligned}y &= 1/2\\ x &= -2z+5/2\\ z &= t\end{aligned} \Rightarrow \begin{bmatrix}5/2\\1/2\\0\end{bmatrix} + \begin{bmatrix}-2\\0\\1\end{bmatrix}t.$$

[you could choose any t here]. Choosing $t=0$, we have $P_1 = \begin{bmatrix}5/2\\1/2\\0\end{bmatrix} \Rightarrow e^{5t}\left(\begin{bmatrix}-2\\0\\1\end{bmatrix}t + \begin{bmatrix}5/2\\1/2\\0\end{bmatrix}\right)$

is a solution.

$\therefore$ The general solution is:

$$X = c_1\begin{bmatrix}-4\\-5\\2\end{bmatrix} + c_2 e^{5t}\begin{bmatrix}-2\\0\\1\end{bmatrix} + c_3 e^{5t}\left(\begin{bmatrix}-2\\0\\1\end{bmatrix}t + \begin{bmatrix}5/2\\1/2\\0\end{bmatrix}\right).$$

3. Find $\mathcal{L}\{(1+e^{2t})^2\}$.

$$\mathcal{L}\{(1+2e^{2t}+e^{4t})\} = \mathcal{L}\{1\} + 2\mathcal{L}\{e^{2t}\} + \mathcal{L}\{e^{4t}\}$$

$$= 1/s + \frac{2}{s-2} + \frac{1}{s-4}.$$

4. Find $\mathcal{L}\{t\}$ using the def^{n} of the Laplace Transform.

$u=t \quad v=-\frac{1}{s}e^{-st}$
$du=dt \quad dv=e^{-st}$

Recall: $\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt$, provided the integral converges.

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\,t\,dt = -\frac{t}{s}e^{-st}\Big|_0^\infty + \int_0^\infty \frac{1}{s}e^{-st}\,dt$$

$\lim\limits_{t\to\infty} \frac{t}{e^{st}}$

$$= -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big|_0^\infty = (0-0)-\left(0-\frac{1}{s^2}\right) = 1/s^2.$$

$= \lim\limits_{t\to\infty} \frac{1}{se^{st}} = 0.$

5. Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s}\right\}$.

$$\frac{1}{s^2+3s} = \frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3} \Rightarrow 1 = A(s+3) + Bs$$

$$\Rightarrow 1 = (A+B)s + 3A \Rightarrow A = -B \;\&\; 3A = 1 \Rightarrow A = 1/3 \;\&\; B = -1/3.$$

So, $\frac{1}{s^2+3s} = \frac{1}{3s} - \frac{1}{3(s+3)}$.

So, $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3s} - \frac{1}{3(s+3)}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$

$$= 1/3 - \frac{e^{-3t}}{3}.$$

6. Use the Laplace Transform to solve the linear IVP $2y' + y = 0, \ y(0) = -3.$

<u>Recall</u>: To solve this we want to $\mathcal{L}$ both sides, isolate for $Y(s)$, then $\mathcal{L}^{-1}$ both sides, using the notation that $\mathcal{L}\{y\} := Y(s)$.

$$\mathcal{L}\{2y' + y\} = \mathcal{L}\{0\} \Leftrightarrow 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 0$$

$$\Leftrightarrow 2[sY(s) - y(0)] + Y(s) = 0$$

$$\Leftrightarrow Y(s)[2s+1] + 2(3) = 0 \Leftrightarrow Y(s) = \frac{-6}{2s+1}$$

$$\Leftrightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{-6}{2s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{-3}{s+\frac{1}{2}}\right\} = -3\mathcal{L}^{-1}\left\{\frac{1}{s+\frac{1}{2}}\right\}$$

$$= -3e^{-t/2}. \quad \therefore \ y(t) = -3e^{-t/2}.$$