Math 2X03: Week #9 Practice Problems

3.5: # 1, 3, 7, 9, 13, 17, 18.

$\boxed{1.}$ Show that the eq^n $x+y-z+\cos(xyz)=0$ can be solved for $z=g(x,y)$ near the origin. Find $\dfrac{\partial g}{\partial x}$ & $\dfrac{\partial g}{\partial y}$ at $(0,0)$.

Recall: [Theorem 11]: Special Implicit Function Theorem:

Suppose $F:\mathbb{R}^{n+1}\to\mathbb{R}$, $(x_1,\dots,x_n,z)\mapsto F(x,z)$, has continuous partial derivatives.
Assume (x_0,z_0) satisfies $F(x_0,z_0)=0$ & $\dfrac{\partial F}{\partial z}(x_0,z_0)\neq 0$.
Then $\exists$ a ball U containing $x_0\in\mathbb{R}^n$ & a nbhd V of $z_0\in\mathbb{R}$ s.t. $\exists!$ function $z=g(x)$ defined for $x\in U$ & $z\in V$ s.t. $F(x, g(x))=0$. If $x\in U$ & $z\in V$ satisfy $F(x,z)=0 \Rightarrow z=g(x)$. & $z=g(x)$ is C^1 with derivative given by:

$$Dg(x) = \frac{-1}{\frac{\partial F}{\partial z}(x,z)} D_xF(x,z)\Big|_{z=g(x)}.$$

Here we have $F:\mathbb{R}^3\to\mathbb{R}$
$(x,y,z)\mapsto x+y-z+\cos(x,y,z)$, &
$(x_0,z_0)=(0,0,z_0)$.

① $DF(x,z) = \left[\dfrac{\partial F}{\partial x}\ \ \dfrac{\partial F}{\partial y}\ \ \dfrac{\partial F}{\partial z}\right] = [1-\sin(xyz)yz \quad 1-\sin(xyz)xz \quad -1-\sin(xyz)xy]$

So, we can see F has continuous partials. Also, $F(x,y,z)=0$ $\forall (x,y,z)\in\mathbb{R}^3$. So, in particular, $F(x_0,z_0)=F(0,0,z_0)=0$.

$$\frac{\partial F}{\partial z}(x_0,z_0)=\frac{\partial F}{\partial z}(0,0,z_0) = -1-\sin(0)\cdot 0 = -1\neq 0.$$

$\therefore$ By Theorem 11 $\exists$ a ball $U\subseteq\mathbb{R}^2$ containing $(0,0)$ &

a nbhd $V \subseteq \mathbb{R}$ containing z_0 s.t. $\exists!$ function $z = g(x)$ for $x \in U$ & $z \in V$ s.t. $F(x, g(x)) = 0$.

$\therefore$ The eq^{n} $x + y - z + \cos(xyz) = 0$ can be solved for $z = g(x,y)$ near $(0,0)$.

Also, by Theorem 11 we have:

$$Dg(x) = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \frac{-1}{\frac{\partial F}{\partial z}(x,z)} \underbrace{D_x F(x,z)}_{\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{bmatrix}} \Bigg|_{z = g(x)}$$

$$\Rightarrow Dg(0,0) = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \Bigg|_{(0,0)} = \frac{-1}{-1} [1 \;\; 1] = [1 \;\; 1].$$

7. Show that $x^3z^2 - z^3yx = 0$ is solvable for z as a function of (x,y) near $(1,1,1)$, but not near the origin. Compute $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ at $(1,1)$.

$F(x,y,z) = x^3z^2 - z^3yx$. $DF = [3x^2z^2 - z^3y \quad -z^3x \quad 2x^3z - 3z^2yx]$

So, F has continuous partials. Here $x_0 = (1,1)$ & $z_0 = 1$. $\frac{\partial F}{\partial z}(1,1,1) = 2-3 = -1 \neq 0$. So, we can write $z = g(x,y)$ for some function g near $(1,1,1)$. However, $\frac{\partial F}{\partial z}(0,0,0) = 0$ $\Rightarrow$ we can't use Theorem 11 to write z as a function of (x,y) near $(0,0,z)$. In particular, in a nbhd of $(0,0)$ we have $x \neq 0$ & $y \neq 0$. Suppose $\exists!$ function g s.t. in a nbhd of $(0,0)$ we have $g(x,y) = z$, & $F(x,y,g(x,y)) = 0$.

$z^2[x^3 - zyx] = 0 \Rightarrow z = 0$ or $x^3 - zyx = 0$. In the piece of the nbhd where $y = x$ we have that this eq'n would be satisfied for $z = 0$ or $z = x$. So, since there is no unique value of z, there can be no unique function in this nbhd s.t. $g(x,y) = z$

[b/c $g(x,x) = 0$ & $g(x,x) = x$ works].

$$Dg(1,1) = \left[\frac{\partial g}{\partial x} \quad \frac{\partial g}{\partial y}\right] = \frac{-1}{\frac{\partial F}{\partial z}(1,1,1)} \left[\frac{\partial F}{\partial x} \quad \frac{\partial F}{\partial y}\right]\Big|_{(1,1,1)}$$

$$= \frac{-1}{-1}\left[2 \quad -1\right] = \left[2 \quad -1\right].$$

17. Consider the eq'n's $\overset{F_1(x,y,u,v)=}{x^2-y^2-u^3+v^2+4=0}$

$\overset{F_2(x,y,u,v)=}{2xy+y^2-2u^2+3v^4+8=0}.$

(a) Show that these eq'n's determine functions $u(x,y)$ & $v(x,y)$ near the pt. $(x,y,u,v)=(2,-1,2,1)$.

$$\Delta = \left.\begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{vmatrix}\right|_{(2,-1,2,1)} = \left.\begin{vmatrix} -3u^2 & 2v \\ -4u & 12v^3 \end{vmatrix}\right|_{(2,-1,2,1)}$$

$$= \begin{vmatrix} -12 & 2 \\ -8 & 12 \end{vmatrix} = \underbrace{-144+16}_{=-128} \neq 0.$$ So, by Theorem 12 these eq'n's determine functions $u(x,y)$ & $v(x,y)$ near $(2,-1,2,1)$.

(b) Compute $\frac{\partial u}{\partial x}$ at $(x,y)=(2,-1)$.

Using implicit differentiation we have: at $(2,-1)$

$2x-3u^2u'+2vv'=0$ & $2y-4uu'+12v^3v'=0 \Rightarrow$ we have $u=2$ & $v=1$:

$$\left.\begin{aligned} 4 - 12u' + 2v' &= 0 \\ -2 - 8u' + 12v' &= 0 \end{aligned}\right\} * -6 \qquad \begin{aligned} -24 + 72u' - \cancel{12v'} &= 0 \\ -2 - 8u' + \cancel{12v'} &= 0 \\ \hline -26 + 64u' &= 0 \end{aligned}$$

$$\Rightarrow \underbrace{u'}_{\frac{\partial u}{\partial x}} = \frac{26}{64} = \frac{13}{32}.$$

10 Is it possible to solve the system of eq'n's