Math 2X03: Week #8 Practice Problems

3.4: # 5, 13, 15, 19, 23, 29, 31

5. Find the extrema of f subject to the stated constraints. $f(x,y) = x$, subject to $x^2 + 2y^2 = 3$.

Recall: Theorem 8: Suppose $f: U^{\subset \mathbb{R}^n} \to \mathbb{R}$ & $g: U^{\subset \mathbb{R}^n} \to \mathbb{R}$ are C^1. Let $x_0 \in U$, $g(x_0) = c$, & S level set for g w/ value c [i.e., $S = \{(x_1, \dots, x_n, g(x_1,\dots,x_n)) \mid g(x_1,\dots,x_n) = c\}$]. Assume $\nabla g(x_0) \neq 0$. If $f|_S$ has a local max./min on S at $x_0 \Rightarrow$ there's a $\lambda \in \mathbb{R}$ s.t. $\nabla f(x_0) = \lambda \nabla g(x_0)$.

Consider $g(x,y) = x^2 + 2y^2 - 3$. S is the level surface $g(x,y) = 0$.

Recall: Theorem 9: If $f|_S$ has a max. or min. at $x_0 \Rightarrow \nabla f(x_0)$ is perpendicular to S at x_0.

$\nabla f = (1, 0)$. $\nabla g = (2x, 4y)$.

$\nabla f(x_0) = \lambda \nabla g(x_0) \Rightarrow (1,0) = \lambda(2x, 4y) \Rightarrow 2x\lambda = 1$ & $4y\lambda = 0$

$\Rightarrow x \neq 0$ & $\lambda \neq 0 \Rightarrow \lambda = \frac{1}{2x} \Rightarrow 4y\left(\frac{1}{2x}\right) = 0 \Rightarrow \frac{4y}{2x} = 0 \Rightarrow y = 0$.

$x^2 + 2y^2 - 3 = 0 \Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$. So we have to consider the points $(\sqrt{3}, 0)$ & $(-\sqrt{3}, 0)$.

Here $\nabla g(x_0) = (2x_0, 4y_0) \neq 0$ on our surface S, since $\nabla g(x_0) = 0 \Rightarrow (x_0, y_0) = (0,0)$, but $(0,0)$ is not on S since $0^2 + 2(0)^2 - 3 \neq 0$.

So, if we have a max. or min. it will come from the set $\{(\sqrt{3},0),(-\sqrt{3},0)\}$. Now we must figure out whether these points are max./min..

Recall: Theorem 7 [Global Existence Theorem for Max. + Min.]: Let D be closed + bounded in $\mathbb{R}^n$ + let $f: D \to \mathbb{R}$ be continuous. Then f assumes its absolute max. + min. values at some points x_0 + x_1 of D.

Here $f|_S : S \to \mathbb{R}$. S is given by $x^2+2y^2=3$ is closed + bounded [it's an ellipse]. $f(x,y)=x$ is a continuous function. $\therefore$ By Theorem 7 we know f does have an absolute max. + min. at some points on S.

$f(\sqrt{3},0)=\sqrt{3}$. $f(-\sqrt{3},0)=-\sqrt{3}$.

So, $(\sqrt{3},0)$ is a maximum + $(-\sqrt{3},0)$ is a minimum.

13. Consider the function $f(x,y)=x^2+xy+y^2$ defined on the unit disc $D=\{(x,y)\mid x^2+y^2\leq 1\}$. Use the method of Lagrange multipliers to locate the max. + min. points of f on the unit circle. Use this to determine the absolute max. + min. values for f on D.

Let $g(x,y)=x^2+y^2-1$.

Recall: Lagrange Multiplier Strategy for Finding Absolute Max. + Min. on Regions w/ Boundary: Let f be differentiable on a closed + bounded region $D=U\cup\partial U$, U open + ∂U smooth. To find the absolute max. + min. of f on D:

(i) Locate all critical points of f in U.
(ii) Use Lagrange multipliers to locate critical pts of $f|_{\partial U}$.
(iii) Compute f at these pts.
(iv) Select largest + smallest.

Here $D = U \cup \partial U$, where $U = \{(x,y) \mid x^2+y^2<1\}$, $\partial U = \{(x,y) \mid x^2+y^2=1\}$. We can realize ∂U as the level surface $g(x,y)=0$.

D is closed & bounded & $f|_D$ is continuous $\Rightarrow$ f assumes its absolute max/min. at some points of D (by Theorem 7).

① Critical points in U:

$\frac{\partial f}{\partial x} = 2x+y$. $\frac{\partial f}{\partial y} = 2y+x$. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow 2x=-y$ & $2y=-x$

$\Rightarrow 2(-2x)=-x \Rightarrow -4x=-x \Rightarrow 4x=x \Rightarrow 4x-x=0 \Rightarrow 3x=0 \Rightarrow x=0$

$\Rightarrow y=0$. $0^2+0^2<1 \Rightarrow (0,0) \in U$.

② Critical pts on ∂U:

$\nabla f = (2x+y,\ 2y+x)$. $\nabla g = (2x, 2y)$.

$\nabla f = \lambda \nabla g \Rightarrow 2x+y = \lambda 2x$ & $2y+x = \lambda 2y$

$\Rightarrow 2x - 2x\lambda + y = 0$ & $2y - \lambda 2y + x = 0 \Rightarrow -2x[1-\lambda] = y$

& $2y[1-\lambda]+x=0 \Rightarrow 2[-2x[1-\lambda]][1-\lambda]+x=0$

$\Rightarrow -4x[1-2\lambda+\lambda^2]+x=0 \Rightarrow x=0$ & $y=0$ or

$\frac{x}{4x} = 1-2\lambda+\lambda^2 \Rightarrow \lambda^2-2\lambda+\frac{3}{4}=0$. $\frac{2 \pm \sqrt{4-4(\frac{3}{4})}}{2}$

$= \frac{2\pm\sqrt{1}}{2} = \frac{3}{2}$ or $\frac{1}{2}$. $\lambda = \frac{3}{2}$ or $\lambda = \frac{1}{2}$

$\Rightarrow y = -2x[1-\frac{3}{2}]$ or $y = -2x[1-\frac{1}{2}] \Rightarrow y = -2x[-\frac{1}{2}]$ or $-2x[\frac{1}{2}]$

$\Rightarrow y = x$ or $y=-x$.

$x^2+y^2=1 \Rightarrow x^2+(\pm x)^2=1 \Rightarrow 2x^2=1 \Rightarrow x^2=\frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$.

So, we have $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, & $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

(iii) $f(0,0) = 0.$

$f(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}.$

$f(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = \frac{1}{2} - \frac{1}{2} + \frac{1}{2} = \frac{1}{2}.$

(iv) So the abs. max. is $\frac{3}{2}$ (at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ & $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$)

& the abs. min is 0 (at $(0,0)$).