21. Find the local maxima & minima for $z = (x^2+3y^2)e^{1-x^2-y^2}$.

$$\frac{\partial f}{\partial x} = 2xe^{1-x^2-y^2} + (x^2+3y^2)[-2x]e^{1-x^2-y^2}$$

$$= 2xe^{1-x^2-y^2}[1-x^2-3y^2] = 0 \Rightarrow x=0 \text{ or } 1-x^2-3y^2=0.$$

$$\frac{\partial f}{\partial y} = 6ye^{1-x^2-y^2} + (x^2+3y^2)[-2y]e^{1-x^2-y^2}$$

$$= 2ye^{1-x^2-y^2}[3-x^2-3y^2] = 0 \Rightarrow y=0 \text{ or } 3-x^2-3y^2=0.$$

If $x=0 \Rightarrow y=0$ or $3-3y^2=0 \Rightarrow 1-y^2=0 \Rightarrow y^2=1 \Rightarrow y=\pm 1$.

If $y=0 \Rightarrow x=0$ or $1-x^2=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$.

If $x \neq 0$ & $y \neq 0$ $\Rightarrow 1-x^2-3y^2=0$ & $3-x^2-3y^2=0$.

$\Rightarrow 2+(1-x^2-3y^2)=0 \Rightarrow 2=0$?!.

So, our critical points are $(0,0)$, $(0,1)$, $(0,-1)$, $(1,0)$, & $(-1,0)$.

$$\frac{\partial^2 f}{\partial x^2} = \left[2e^{1-x^2-y^2} + 2x[-2x]e^{1-x^2-y^2}\right][1-x^2-3y^2] + 2xe^{1-x^2-y^2}\cdot[-2x].$$

$$\frac{\partial^2 f}{\partial y^2} = \left[2e^{1-x^2-y^2} + 2y[-2y]e^{1-x^2-y^2}\right][3-x^2-3y^2] + 2ye^{1-x^2-y^2}[-6y].$$

$$\frac{\partial^2 f}{\partial x\partial y} = 2x[-2y]e^{1-x^2-y^2}[1-x^2-3y^2] + 2xe^{1-x^2-y^2}[-6y].$$

$(0,0)$: $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = 2e^1 > 0$. $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = 2e^1\cdot 3 = 6e^1$. $\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(0,0)} = 0$.

$D = (2e^1)(6e^1) - 0^2 = 12e^2 > 0$. So, $(0,0)$ is a rel. min..

$(0,\pm1)$ $\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,\pm1)} = 2e^0\cdot[1-3] = -4 < 0$. $\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,\pm1)} = [2e^0 - 4e^0][3-3] - 12e^0 = -12$

$\left.\frac{\partial^2 f}{\partial x\partial y}\right|_{(0,\pm1)} = 0$. $D = (-4)(-12) - 0^2 > 0$. $\therefore$ Both $(0,1)$ & $(0,-1)$ are local max..

$(\pm 1, 0)$: $\frac{\partial^2 f}{\partial x^2}\Big|_{(\pm 1, 0)} = -4.$ $\frac{\partial^2 f}{\partial y^2}\Big|_{(\pm 1, 0)} = 2 \cdot 2 = 4.$ $\frac{\partial^2 f}{\partial x \partial y}\Big|_{(\pm 1, 0)} = 0.$

$D = (-4)(4) - 0^2 < 0.$ So, $(1,0)$ & $(-1,0)$ are saddle points.

25. Let $f(x,y) = x^2 - 2xy + y^2$. Here $D = 0$. Can you say whether the critical points are local minima, local maxima, or saddle points?

$D = 0 \Rightarrow$ the 2^{nd}-derivative max.-min. test doesn't tell us anything, but we can still find the critical points & try to examine them using other methods:

$\frac{\partial f}{\partial x} = 2x - 2y$, $\frac{\partial f}{\partial y} = -2x + 2y$. $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow 2x = 2y$ & $2x = 2y$

$\Rightarrow x = y$. So all points of the form (x, x) are critical points (i.e. all points along the line $y = x$).

$f(x,x) = x^2 - 2x^2 + x^2 = 0.$

Notice: $f(x,y) = \underbrace{(x-y)^2}_{\geq 0} = (x-y)(x-y) = x^2 - 2xy + y^2 \geq 0.$

$\therefore$ All points not on $y = x$ are $\geq$ all points on the line $y = x$
$\Rightarrow$ the critical points (x, x) are all local minima.

27. Suppose $f: \mathbb{R}^3 \to \mathbb{R}$ is C^2 & that x_0 is a critical point for f. Suppose $Hf(x_0)(h) = h_1^2 + h_2^2 + h_3^2 + 4h_2h_3$. Does f have a local max, min., or saddle at x_0?

Recall: • $Hf(x_0)(h)$ is pos. definite if $Hf(x_0)(h) \geq 0 \ \forall \ h \in \mathbb{R}^3$ & $Hf(x_0)(h) = 0 \Leftrightarrow h = 0$.

$Hf(x_0)(h)$ is neg. definite if $Hf(x_0)(h) \leq 0 \ \forall \ h \in \mathbb{R}^3$ & $Hf(x_0)(h) = 0 \Leftrightarrow h = 0$.

Here $Hf(x_0)(h) = \frac{1}{2} [h_1 \ h_2 \ h_3] \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \vdots & & \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_3 \partial x_3} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$ is the Hessian of f at x_0.

Theorem 5 [2nd Derivative Test For Local Extrema]: If $f: U \subset \mathbb{R}^n \to \mathbb{R}$ is C^3, $\vec{x}_0 \in U$ a critical point of f, & $Hf(x_0)$ pos. def. $\Rightarrow$ x_0 rel. min..

" & $Hf(x_0)$ neg. def. $\Rightarrow$ x_0 rel. max..

If $Hf(x_0)$ is neither pos. nor neg. def. $\Rightarrow$ x_0 is a saddle point.

Here $Hf(x_0)$ is not pos. def. & not neg. def.. Indeed,

$Hf(x_0)((\sqrt{2},-1,1)) = (\sqrt{2})^2 + (-1)^2 + 1^2 + 4(-1)(1) = 2+1+1-4=0.$

But $h=(\sqrt{2},-1,1) \neq 0$.

$\therefore$ x_0 is a saddle point.