3.2:5. Determine the $2^{nd}$ order Taylor Formula for $F(x,y) = e^{x+y}$ about the point $(0,0)$.

Recall: Second-Order Taylor Formula: Let $F: U \subset \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives of third order. Then we may write

$$F(x_0 + h) = F(x_0) + \sum_{i=1}^{n} h_i \frac{\partial F}{\partial x_i}(x_0) + \frac{1}{2} \underbrace{\sum_{i,j=1}^{n} h_i h_j}_{n^2 \text{ terms}} \frac{\partial^2 F}{\partial x_i \partial x_j}(x_0)$$

$$+ R_2(x_0, h), \quad \text{where } \frac{R_2(x_0,h)}{\|h\|^2} \to 0 \text{ as } h \to 0.$$

$F(0,0) = e^0 = 1.$ Here $F: \mathbb{R}^2 \to \mathbb{R}$, so $n = 2$.

$x_1 := x.$ $x_2 := y.$

$$\left.\frac{\partial F}{\partial x}\right|_{(0,0)} = e^{x+y} = 1. \quad \left.\frac{\partial F}{\partial y}\right|_{(0,0)} = e^{x+y} = 1. \quad \left.\frac{\partial^2 F}{\partial x^2}\right|_{(0,0)} = \left.\frac{\partial^2 F}{\partial y^2}\right|_{(0,0)} = \left.\frac{\partial F}{\partial x \partial y}\right|_{(0,0)} = \left.e^{x+y}\right|_{(0,0)} = 1.$$

$$\frac{1}{2}\begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 F}{\partial x_1^2} & \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ \frac{\partial^2 F}{\partial x_2 \partial x_1} & \frac{\partial^2 F}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$F((0,0) + (h_1,h_2)) = 1 + \sum_{i=1}^{2} h_i \frac{\partial F}{\partial x_i}(0,0) + \frac{1}{2} \sum_{i,j=1}^{2} h_i h_j \frac{\partial^2 F}{\partial x_i \partial x_j}(0,0) + R_2(0,h)$$

$$= 1 + h_1 + h_2 + \frac{1}{2}[h_1^2 + h_1 h_2 + h_2^2 + h_2 h_1] + R_2(h_1,h_2)$$

$$= 1 + h_1 + h_2 + \frac{h_1^2}{2} + h_1 h_2 + \frac{h_2^2}{2} + R_2(h_1,h_2).$$

9. Calculate the second-order Taylor approximation to $F(x,y) = \cos x \sin y$ at the point $(\pi, \frac{\pi}{2})$.

$F(\pi, \frac{\pi}{2}) = \cos\pi \sin\frac{\pi}{2} = [-1][1] = -1.$

$$\left.\frac{\partial F}{\partial x}\right|_{(\pi,\frac{\pi}{2})} = -\sin(x)\sin(y)\Big|_{(\pi,\frac{\pi}{2})} = 0. \quad \left.\frac{\partial F}{\partial y}\right|_{(\pi,\frac{\pi}{2})} = \cos x \cos y\Big|_{(\pi,\frac{\pi}{2})} = 0.$$

$$\left.\frac{\partial^2 F}{\partial x^2}\right|_{(\pi,\frac{\pi}{2})} = -\cos(x)\sin(y)\Big|_{(\pi,\frac{\pi}{2})} = -[-1][1] = 1.$$

$$\left.\frac{\partial^2 F}{\partial x \partial y}\right|_{(\pi,\frac{\pi}{2})} = -\sin(x)\cos(y)\Big|_{(\pi,\frac{\pi}{2})} = 0. \quad \left.\frac{\partial^2 F}{\partial y^2}\right|_{(\pi,\frac{\pi}{2})} = \cos x[-\sin y]\Big|_{(\pi,\frac{\pi}{2})} = [-1][-1] = 1.$$

let $h_1 = (x - \pi)$ & $h_2 = (y - \frac{\pi}{2})$:

$$F(\pi, \tfrac{\pi}{2}) + \sum_{i=1}^{2} h_i \frac{\partial F}{\partial x_i}(\pi, \tfrac{\pi}{2}) + \frac{1}{2} \sum_{i,j=1}^{2} h_i h_j \frac{\partial^2 F}{\partial x_i \partial x_j}(\pi, \tfrac{\pi}{2})$$

$$= -1 + h_1 \cdot (0) + h_2 \cdot (0) + \frac{1}{2} \cdot \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$= -1 + \frac{1}{2} \begin{bmatrix} h_1 & h_2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = -1 + \frac{1}{2} \left[ h_1^2 + h_2^2 \right]$$

$$= -1 + \frac{1}{2}(x - \pi)^2 + \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2.$$

#5. Approximate $f(-0.1,-0.1)$.

$f(-0.1,-0.1) \approx 0.8187.$

$h_1 = h_2 = -0.1.$ $1 + \left(-\frac{1}{10}\right) + \left(-\frac{1}{10}\right) + \frac{\left(-\frac{1}{10}\right)^2}{2} + \left(-\frac{1}{10}\right)\left(-\frac{1}{10}\right) + \frac{\left(-\frac{1}{10}\right)^2}{2}$

$= 1 - \frac{2}{10} + \frac{1}{200} + \frac{1}{100} + \frac{1}{200} = \frac{4}{200} + \frac{200}{200} - \frac{40}{200}$

$= \frac{164}{200} = \frac{82}{100} = \frac{41}{50} = 0.82.$

[Close b/c $(-0.1,-0.1)$ pretty close to $(0,0)$.]