

3.2:5. Determine the 2nd order Taylor formula for $f(x,y) = e^{x+y}$ about the point $(0,0)$.

Recall: Second-order Taylor Formula: Let $F: U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ have continuous partial derivatives of third order. Then we may write

$$F(x_0+h) = F(x_0) + \sum_{i=1}^n h_i \frac{\partial F}{\partial x_i}(x_0) + \frac{1}{2} \sum_{i,j=1}^n h_i h_j \frac{\partial^2 F}{\partial x_i \partial x_j}(x_0) + R_2(x_0, h)$$

where $\frac{R_2(x_0, h)}{\|h\|^2} \rightarrow 0$ as $h \rightarrow 0$.

$F(0,0) = e^0 = 1$. Here $F: \mathbb{R}^2 \rightarrow \mathbb{R}$, so $n=2$.

$x_1 := x, x_2 := y$

$$\frac{\partial F}{\partial x} \Big|_{(0,0)} = e^{x+y} = 1, \quad \frac{\partial F}{\partial y} \Big|_{(0,0)} = e^{x+y} = 1, \quad \frac{\partial^2 F}{\partial x^2} \Big|_{(0,0)} = \frac{\partial^2 F}{\partial y^2} \Big|_{(0,0)} = \frac{\partial F}{\partial x \partial y} \Big|_{(0,0)} = e^{x+y} \Big|_{(0,0)} = 1.$$

$$F(0,0) + (h_1, h_2) = 1 + \sum_{i=1}^2 h_i \frac{\partial F}{\partial x_i}(0,0) + \frac{1}{2} \sum_{i,j=1}^2 h_i h_j \frac{\partial^2 F}{\partial x_i \partial x_j}(0,0) + R_2(h_1, h_2)$$

$$= 1 + h_1 + h_2 + \frac{1}{2} [h_1^2 + h_1 h_2 + h_2^2 + h_2 h_1] + R_2(h_1, h_2)$$

$$= 1 + h_1 + h_2 + \frac{h_1^2}{2} + h_1 h_2 + \frac{h_2^2}{2} + R_2(h_1, h_2).$$

9. Calculate the second-order Taylor approximation to $f(x,y) = \cos x \sin y$ at the point $(\pi, \frac{\pi}{2})$.

$$f(\pi, \frac{\pi}{2}) = \cos \pi \sin \frac{\pi}{2} = [-1][1] = -1.$$

$$\frac{\partial f}{\partial x} \Big|_{(\pi, \frac{\pi}{2})} = -\sin(x) \sin(y) \Big|_{(\pi, \frac{\pi}{2})} = 0, \quad \frac{\partial f}{\partial y} \Big|_{(\pi, \frac{\pi}{2})} = \cos(x) \cos(y) \Big|_{(\pi, \frac{\pi}{2})} = 0.$$

$$\frac{\partial^2 f}{\partial x^2} \Big|_{(\pi, \frac{\pi}{2})} = -\cos(x) \sin(y) \Big|_{(\pi, \frac{\pi}{2})} = -[-1][1] = 1.$$

$$\frac{\partial^2 f}{\partial x \partial y} \Big|_{(\pi, \frac{\pi}{2})} = -\sin(x) \cos(y) \Big|_{(\pi, \frac{\pi}{2})} = 0, \quad \frac{\partial^2 f}{\partial y^2} \Big|_{(\pi, \frac{\pi}{2})} = \cos(x) [-\sin(y)] \Big|_{(\pi, \frac{\pi}{2})} = [-1][-1] = 1.$$

$$f(\pi, \frac{\pi}{2}) + \sum_{i=1}^2 h_i \frac{\partial f}{\partial x_i}(\pi, \frac{\pi}{2}) + \frac{1}{2} \sum_{i,j=1}^2 h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\pi, \frac{\pi}{2})$$

$$f(x, y) = -1 + h_1 \cdot 1 + h_2 \cdot 1 + \frac{1}{2} [h_1 \ h_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$$= -1 + \frac{1}{2} [h_1 \ h_2] \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = -1 + \frac{1}{2} [h_1^2 + h_2^2]$$

$$= -1 + \frac{1}{2} (x - \pi)^2 + \frac{1}{2} (y - \frac{\pi}{2})^2$$

11. Let $g(x, y) = \sin(x) - 3x^2 \log y + 1$. Find the degree 2 polynomial which best approximates g near the point $(\frac{\pi}{2}, 1)$.

$$g(\frac{\pi}{2}, 1) = -3 \cdot \frac{\pi^2}{4} \log(1) + 1 = 1 = h_1 = 2.$$

$$\frac{\partial g}{\partial x} \Big|_{(\frac{\pi}{2}, 1)} = \cos(x) - 6x \log y = -6 \log(1) = 0.$$

$$\frac{\partial g}{\partial y} \Big|_{(\frac{\pi}{2}, 1)} = \sin(x) - \frac{3x^2}{y} = \sin(\frac{\pi}{2}) - \frac{3(\frac{\pi}{2})^2}{1} = \frac{1}{2} - \frac{3\pi^2}{4} = \frac{1 - 3\pi^2}{2}$$

$$\frac{\partial^2 g}{\partial x^2} \Big|_{(\frac{\pi}{2}, 1)} = -\sin(x) - 6 \log y = -1 - 0 = -1$$

$$\frac{\partial^2 g}{\partial x \partial y} \Big|_{(\frac{\pi}{2}, 1)} = -\frac{3x^2}{y^2} = -\frac{3(\frac{\pi}{2})^2}{1} = -\frac{3\pi^2}{4}$$

$$\frac{\partial^2 g}{\partial y^2} \Big|_{(\frac{\pi}{2}, 1)} = \frac{6x^2}{y^3} = \frac{6(\frac{\pi}{2})^2}{1} = \frac{3\pi^2}{2}$$

$$g(x, y) \approx 1 + 0(x - \frac{\pi}{2}) + \frac{1}{2}(-1)(x - \frac{\pi}{2})^2 + \frac{1}{2}(\frac{3\pi^2}{2})(y - 1)^2 - \frac{3\pi^2}{4}(x - \frac{\pi}{2})(y - 1)$$

$$= 1 - \frac{1}{2}(x - \frac{\pi}{2})^2 + \frac{3\pi^2}{4}(y - 1)^2 - \frac{3\pi^2}{4}(x - \frac{\pi}{2})(y - 1)$$

$$= 1 - \frac{1}{2}x^2 + \frac{\pi x}{2} - \frac{\pi^2}{8} + \frac{3\pi^2}{4}y^2 - \frac{3\pi^2}{2}y + \frac{3\pi^3}{4} - \frac{3\pi^2}{4}x + \frac{3\pi^3}{4} - \frac{3\pi^2}{4}y + \frac{3\pi^3}{4}$$

#5.

Approximate
 $F(-0.1, -0.1)$.

$$F(-0.1, -0.1) \approx 0.8182.$$

$$h_1 = h_2 = -0.1. \quad 1 + \left(-\frac{1}{10}\right) + \left(-\frac{1}{10}\right) + \frac{\left(-\frac{1}{10}\right)^2}{2} + \left(-\frac{1}{10}\right)\left(-\frac{1}{10}\right) + \frac{\left(-\frac{1}{10}\right)^2}{2}$$

$$= 1 - \frac{2}{10} + \frac{1}{200} + \frac{1}{100} + \frac{1}{200} = \frac{4}{200} + \frac{300}{200} = \frac{304}{200}$$

$$= \frac{164}{100} = \frac{82}{50} = \frac{41}{25} = 0.82.$$

[Close b/c $(-0.1, -0.1)$ pretty close to $(0, 0)$.]