# Math 2X03: Week #5 Practice Problems

2.5: # 30, 33, 35
2.6: # 3ⓐⓒ, 5ⓑ, 7, 9ⓒ, 13, 15, 17ⓒ, 19, 21, 25.

2.5:

30. For what integers $p > 0$ is $F(x) = \begin{cases} x^p \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$ differentiable? For what $p$ is the derivative continuous?

$$F'(x) = p x^{p-1} \sin(1/x) + x^p \cos(1/x) \cdot \left[ -\frac{1}{x^2} \right]$$

$$= p x^{p-1} \sin(1/x) - x^{p-2} \cos(1/x).$$

- By def'n a function is "differentiable" if it is differentiable at every point in its domain.

If $x \neq 0$ we can see that $F$ will be differentiable. So, let's see what happens at $x = 0$:

$$F'(0) = \lim_{h \to 0} \frac{F(0+h) - F(0)}{h} = \lim_{h \to 0} \frac{h^p \sin(\frac{1}{h}) - 0}{h}.$$

As $h \to 0$ $\sin(1/h)$ keeps oscillating b/w $-1 \leq \sin(1/h) \leq 1$.
So if $p = 1$ we can see $\lim_{h \to 0} \frac{\cancel{h} \sin(1/h)}{\cancel{h}}$ does not



x sin(1/x)

exist. However, if $p$ is an integer s.t. $p > 1$
$\Rightarrow p - 1 > 0 \Rightarrow \lim_{h \to 0} \frac{h^p \sin(1/h)}{h} = \lim_{h \to 0} h^{p-1} \sin(1/h) = 0.$
$\therefore F$ is differentiable for integers $p > 1$.

- If $x \neq 0$ we can see $F'(x)$ is continuous for all $p > 0$, so let's see what happens at $x = 0$:

We want to find the $p > 0$ s.t. $\lim_{x \to 0} \left[ p x^{p-1} \sin(1/x) - x^{p-2} \cos(1/x) \right] =$

Again, $\sin(1/x)$ & $\cos(1/x)$ keep oscillating as $x \to 0$, so if $p=1$ we can see that this limit won't exist: $\lim_{x\to 0} \sin(1/x) - x^{-1}\cos(1/x)$ ~~not~~ exist.

& is $p=2$ the limit does not exist: $\lim_{x\to 0} 2x\sin(1/x) - \cos(1/x)$.

However, if $p>2$ is an integer, then

$$\lim_{x\to 0} p\,x^{p-1}\sin(1/x) - x^{p-2}\cos(1/x) = 0.$$

So, the derivative is continuous for integers $p>2$.

Notice: When $p=2$ are function is:

$$F(x) = \begin{cases} x^2\sin(1/x) & x\neq 0 \\ 0 & x=0. \end{cases}$$

$x^2\sin(1/x)$

We just showed that this function is differentiable but it's derivative is not continuous [i.e., it's differentiable but not $C^1$ (not continuously differentiable)].

So, we know continuous partials $\overset{\text{Theorem 9 [pg. 115]}}{\Longrightarrow}$ differentiable, but

differentiable $\not\Longrightarrow$ continuous partials, &

partials not continuous $\not\Longrightarrow$ not differentiable.

33. Let $F:\mathbb{R}^4 \to \mathbb{R}$ & $c(t):\mathbb{R}\to\mathbb{R}^4$. Suppose $DF(1,1,\pi,e^6) = (0,1,3,-7)$, $c(\pi) = (1,1,\pi,e^6)$ & $c'(\pi) = (19,11,0,1)$. Find $\frac{d(F\circ c)}{dt}$ when $t=\pi$.

$$\frac{d}{dt}(F\circ c)(\pi) = F'(c(\pi))\cdot c'(\pi) = F'(1,1,\pi,e^6)\cdot c'(\pi)$$

$$= \begin{bmatrix} 0 & 1 & 3 & -7 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \\ 0 \\ 1 \end{bmatrix} = 11-7 = 4.$$

35. If $z = f(x-y)$, use the chain rule to show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial x} = \frac{\partial f}{\partial g}(1) = \frac{\partial f}{\partial g}.$$

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial y} = \frac{\partial f}{\partial g}(-1) = -\frac{\partial f}{\partial g}.$$

So, $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{\partial f}{\partial g} - \frac{\partial f}{\partial g} = 0.$ ✓

2.6:

3. Compute the directional derivatives of the following along unit vectors at the indicated points in directions parallel to the given vector:

Recall: If $f: \mathbb{R}^3 \to \mathbb{R}$, the directional derivative of $f$ at $x$ along the unit vector $v$ is given by

$$\nabla f \cdot v = \frac{d}{dt} f(x+tv)\Big|_{t=0},$$ if this exists.

(a) $f(x,y) = x^y$, $(x_0, y_0) = (e,e)$, $d = 5i + 12j$.

$d = 5i + 12j = (5, 12)$ is not a unit vector since $\sqrt{5^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13 \neq 1$.

So, we should instead use the vector $\left(\frac{5}{13}, \frac{12}{13}\right)$ b/c this is a unit vector parallel to $(5,12)$.

$$\nabla f(e,e) \cdot \left(\frac{5}{13}, \frac{12}{13}\right) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\Big|_{(e,e)} \left(\frac{5}{13}, \frac{12}{13}\right)$$

$\frac{d}{dx} a^x = (\ln a) a^x$ where $a$ is a constant.

$$= \left(y x^{y-1}, (\ln x) x^y\right)\Big|_{(e,e)} \cdot \left(\frac{5}{13}, \frac{12}{13}\right)$$

$$= \left(e^e, e^e\right) \cdot \left(\frac{5}{13}, \frac{12}{13}\right) = \frac{5}{13}e^e + \frac{12}{13}e^e = \frac{17}{13}e^e.$$

(c) $F(x,y,z) = xyz$, $(x_0, y_0, z_0) = (1,0,1)$, $d = (1,0,-1)$.

$\|(1,0,-1)\| = \sqrt{2}$. $\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$.

$\nabla F(1,0,1) \cdot \frac{1}{\sqrt{2}}(1,0,-1) = (yz, xz, xy)\Big|_{(1,0,1)} \cdot \frac{1}{\sqrt{2}}(1,0,-1)$

$= (0,1,0) \cdot \frac{1}{\sqrt{2}}(1,0,-1) = 0.$

5. (b) Let $F(x,y,z) = x^3 - y^3 + z^3$. Find the maximum value for the directional derivative of $F$ at the point $(1,2,3)$.

To find the maximal value for the directional derivative we need to choose the unit vector $v$ which points in the direction along which $F$ is increasing the fastest. By Theorem 13 we know this is $\nabla F(1,2,3)$.

$\nabla F(1,2,3) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Big|_{(1,2,3)} = (3x^2, -3y^2, 3z^2)\Big|_{(1,2,3)}$

$= (3, -12, 27)$.

$\|\nabla F(1,2,3)\| = \sqrt{3^2 + (12)^2 + 27^2} = \sqrt{9 + 144 + 729} = \sqrt{882}$

The directional derivative of $F$ at $(1,2,3)$ along $\frac{1}{\sqrt{882}}(3,-12,27)$ is:

$\nabla F(1,2,3) \cdot (3,-12,27) \cdot \frac{1}{\sqrt{882}} = \frac{882}{\sqrt{882}} = \sqrt{882} = \sqrt{21^2 \cdot 2} = 21\sqrt{2}$.

So, the maximum value for the directional derivative of $F$ is $\|\nabla F(1,2,3)\|$.

$$\begin{array}{r} 27 \\ \times 27 \\ \hline 189 \\ 540 \\ \hline 729 \\ 144 \\ \hline 873 \\ 9 \\ \hline 882 \end{array}$$

$$\begin{array}{r} 21 \\ \times 21 \\ \hline 21 \\ 420 \\ \hline 441 \\ \times 2 \\ \hline 882 \end{array}$$

**Note:** $(0,0,1)$ is also a unit normal to the surface.

15. Show that the "Tangent plane to level surfaces" definition yields, as a special case, the formula for the plane tangent to the graph of $f(x,y)$ by regarding the graph as a level surface of $F(x,y,z) = f(x,y) - z$.

**Recall:** For $f: \mathbb{R}^2 \to \mathbb{R}$, the tangent plane of the graph of $f$ at the point $(x_0, y_0, f(x_0,y_0))$ is defined by the eq$^{n}$ $z = f(x_0,y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)$. [Pg. 110]

The graph of $f(x,y)$: $z = f(x,y)$ is the level surface $F(x,y,z) = 0$. Using our "tangent plane to surfaces" def$^{n}$, the tangent plane to this level surface is $\nabla F(x_0,y_0,z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0$

$$\nabla F(x_0,y_0,z_0) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)\Big|_{(x_0,y_0,z_0)} = \left(\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0), -1\right).$$

So, $\nabla F(x_0,y_0,z_0) \cdot (x-x_0, y-y_0, z-z_0)$

$= \left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0) - z + z_0$. But $z = f(x,y)$

$\Rightarrow z_0 = f(x_0,y_0)$. So $\nabla F(x_0,y_0,z_0) \cdot (x-x_0, y-y_0, z-z_0) = 0$ becomes

$$\left[\frac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0) + f(x_0,y_0) = z,$$

So we can see that the 2 def$^{n}$s coincide.