21. Suppose that a particle following a given path $c(t) = (4e^t, 6t^4, \cos t)$ flies off on a tangent at $t=0$. Compute the position of the particle at the given time $t_1 = 1$.

At $t=0$ the particle is at $c(0) = (4, 0, 1)$. The eq'n of the tangent line at the point $c(0)$ is
$\ell(t) = c(0) + (t-0)\,c'(0) = (4, 0, 1) + t(4e^0, 24(0)^3, -\sin(0))$

$= (4,0,1) + t(4,0,0).$

At $t=1$ the position on this line is:

$\ell(1) = (4,0,1) + (4,0,0) = (8,0,1).$

So the particle is at the position $(8,0,1)$ at time $t_1 = 1$.

22. The position vector for a particle moving on a helix

2.5:

3.ⓒ Verify the first special case of the chain rule for the composition $F \circ c$ where $F(x,y) = (x^2+y^2)\log\sqrt{x^2+y^2}$, $c(t) = (e^t, e^{-t})$.

Recall: • Theorem 11: [Chain Rule]: Let $U \subset \mathbb{R}^n$ & $V \subset \mathbb{R}^m$ be open sets. Let $g: U \subset \mathbb{R}^n \to \mathbb{R}^m$ & $F: V \subset \mathbb{R}^m \to \mathbb{R}^p$ be given functions s.t. g maps U into V, so that $F \circ g$ is defined. Suppose g is differentiable at x_0 & F is differentiable at $g(x_0)$. Then $F \circ g$ is differentiable at x_0 &

$$D(F\circ g)(x_0) = DF(g(x_0))\, Dg(x_0).$$

• First Special Case of Chain Rule: Suppose $c: \mathbb{R} \to \mathbb{R}^3$ is a differentiable path ($c(t) = (x(t), y(t), z(t))$) and $F: \mathbb{R}^3 \to \mathbb{R}$. Let $h(t) = F(c(t)) = F(x(t), y(t), z(t))$. Then

$$\frac{dh}{dt} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial t}.$$

Notice: $\underbrace{\nabla F(c(t))}_{DF(c(t))} \cdot \underbrace{c'(t)}_{Dc(t)} = \begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{bmatrix}\Bigg|_{c(t)} \cdot \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{bmatrix}\Bigg|_{t} = \frac{\partial h}{dt}.$

First special Case of Chain Rule:

So, here $h(t) = F(c(t)) = F(\underbrace{e^t}_{x(t)}, \underbrace{e^{-t}}_{y(t)})$.

$$\frac{dh}{dt} = \frac{dF}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t}$$

$$= 2x\log\sqrt{x^2+y^2} + (x^2+y^2)\frac{\cancel{1}}{(x^2+y^2)^{1/2}}\cancel{\tfrac12}(x^2+y^2)^{-1/2}\cdot 2x\,\Big|_{c(t)} \cdot e^t$$

$$+\, 2y\log\sqrt{x^2+y^2} + (x^2+y^2)\cdot\frac{1}{2(x^2+y^2)}\cdot 2y\,\Big|_{c(t)} \cdot -e^{-t}$$

$$= \left[2[e^{t}] \log\sqrt{e^{2t}+e^{-2t}} + e^{t}\right]e^{t} + \left[2 e^{-t} \log\sqrt{e^{2t}+e^{-2t}} + e^{-t}\right] - e^{-t}$$

$$= e^{2t}\left[2\log\left((e^{2t}+e^{-2t})^{\frac{1}{2}}\right)+1\right] - e^{-2t}\left[2\log\left((e^{2t}+e^{-2t})^{\frac{1}{2}}\right)+1\right]$$

$$= \left[e^{2t} - e^{-2t}\right]\left[\log(e^{2t}+e^{-2t})+1\right].$$

Direct Computation:

$$F\circ c(t) = \left[e^{2t}+e^{-2t}\right]\log\left[(e^{2t}+e^{-2t})^{\frac{1}{2}}\right].$$

$$(F\circ c)'(t) = \left[2e^{2t} - 2e^{-2t}\right]\log\left[(e^{2t}+e^{-2t})^{\frac{1}{2}}\right] + \left[e^{2t}+e^{-2t}\right]\frac{\left[e^{2t}+e^{-2t}\right]^{-\frac{1}{2}}\cdot\left[2e^{2t}-2e^{-2t}\right]}{2(e^{2t}+e^{-2t})^{\frac{1}{2}}}$$

$$= \left[e^{2t}-e^{-2t}\right]\log(e^{2t}+e^{-2t}) + \left[e^{2t}-e^{-2t}\right]$$

$$= \left[e^{2t}-e^{-2t}\right]\left[\log(e^{2t}+e^{-2t})+1\right].$$

7. Let $F(u,v) = (\tan(u-1)-e^{v}, u^2-v^2)$ & $g(x,y) = (e^{x-y}, x-y)$.
Calculate $F\circ g$ & $D(F\circ g)(1,1)$.

$$F\circ g(x,y) = \left(\tan(e^{x-y}-1) - e^{x-y},\ e^{2(x-y)} - (x-y)^2\right).$$

Using Chain Rule:

$$D(F\circ g)(1,1) = DF(g(1,1))\cdot Dg(1,1)$$

$$= \begin{bmatrix} \sec^2(u-1) & -e^{v} \\ 2u & -2v \end{bmatrix}\Bigg|_{(1,0)} \cdot \begin{bmatrix} e^{x-y} & -e^{x-y} \\ 1 & -1 \end{bmatrix}\Bigg|_{(1,1)}$$

$$= \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}.$$

Direct Computation: $(F \circ g)'(1,1) = \begin{bmatrix} \sec^2(e^{x-y}-1)\cdot e^{x-y} - e^{x-y} & \sec^2(e^{x-y}-1)\cdot -e^{x-y} + e^{x-y} \\ 2e^{2(x-y)} - 2(x-y) & -2e^{2(x-y)} + 2(x-y) \end{bmatrix}_{(1,1)}$

$$= \begin{bmatrix} \sec^2(0) - 1 & \sec^2(0)\cdot[-1] + 1 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}.$$

15. Let $F: \mathbb{R}^2 \to \mathbb{R}^2$ $(x,y) \mapsto (e^{x+y}, e^{x-y})$. Let $c(t)$ be a path with $c(0)=(0,0)$ & $c'(0) = (1,1)$.
What is the tangent vector to the image of $c(t)$ under F at $t=0$?

$$(F\circ c)'(0) = DF(c(0))\, Dc(0) = DF(0,0) \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^{x-y} & -e^{x-y} \end{bmatrix}\Bigg|_{(0,0)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

21. Dieterici's eq'n of state for a gas is $P(V-b)e^{\frac{a}{RVT}} = RT$, where a, b, & R are constants. Regard volume V as a function of temperature T & pressure P, & prove that

$$\frac{\partial V}{\partial T} = \frac{R + \frac{a}{TV}}{\frac{RT}{V-b} - \frac{a}{V^2}}.$$

Consider $G(\overset{T}{\overset{\shortparallel}{t}}(T,P), \overset{P}{\overset{\shortparallel}{p}}(T,P), V(T,P)) = (PV - Pb)e^{\frac{a}{RVT}} - RT = 0.$

$$0 = \frac{\partial G}{\partial T} = \frac{\partial G}{\partial t}\underbrace{\frac{\partial t}{\partial T}}_{=1} + \frac{\partial G}{\partial p}\underbrace{\frac{\partial p}{\partial P}}_{=0} + \frac{\partial G}{\partial V}\frac{\partial V}{\partial T} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial V}\frac{\partial V}{\partial T}$$

$$\Rightarrow \frac{\partial V}{\partial T} = -\frac{\frac{\partial G}{\partial t}}{\frac{\partial G}{\partial V}}.$$

$$\frac{\partial G}{\partial t} = \underline{(PV - Pb)}\frac{a}{RV}\frac{-1}{T^2}\underline{e^{\frac{a}{RVT}}} - R = \frac{-a}{RVT^2}RT - R = \frac{-a}{VT} - R.$$

$$\frac{\partial G}{\partial V} = Pe^{\frac{a}{RVT}} + \underline{(PV - Pb)}\frac{a}{RT}\frac{-1}{V^2}\underline{e^{\frac{a}{RVT}}} = Pe^{\frac{a}{RVT}} - \frac{a}{RTV^2}RT$$

$$= Pe^{\frac{a}{RVT}} - \frac{a}{V^2} = \frac{RT}{V-b} - \frac{a}{V^2}.$$

$$\therefore \frac{\partial V}{\partial T} = \frac{-\frac{\partial G}{\partial T}}{\frac{\partial G}{\partial V}} = \frac{R + \frac{a}{TV}}{\frac{RT}{V-b} - \frac{a}{V^2}}.$$

• Prove $\lim_{(x,y)\to(0,0)} \overbrace{\frac{2x^2 + 3y^2}{\sqrt{x^2+y^2}}}^{F(x,y)} = 0$ [using $\varepsilon - \delta$].

We want to show for every $\varepsilon > 0$ $\exists\ \delta > 0$ s.t. $\|\vec{x} - (0,0)\| < \delta \Rightarrow \|F(x) - 0\| < \varepsilon$.

$$|F(x,y)| = \frac{2x^2 + 3y^2}{\sqrt{x^2+y^2}} \leq \frac{3x^2 + 3y^2}{(x^2+y^2)^{1/2}} = \frac{3(x^2+y^2)}{(x^2+y^2)^{1/2}} = 3\sqrt{x^2+y^2}.$$

$\|(x,y) - (0,0)\| = \sqrt{x^2+y^2} < \delta$.

Let's choose $\delta = \frac{\varepsilon}{3}$.

Then $\|(x,y) - (0,0)\| < \frac{\varepsilon}{3} \Rightarrow \|F(x,y) - 0\| = F(x,y) \leq 3\underbrace{\sqrt{x^2+y^2}}_{< \frac{\varepsilon}{3}}$

$< 3\frac{\varepsilon}{3} = \varepsilon$. ✓