5. Find $\frac{\partial w}{\partial x}$ & $\frac{\partial w}{\partial y}$:

(a) $w = xe^{x^2+y^2}$.

$$\frac{\partial w}{\partial x} = e^{x^2+y^2} + x \cdot [2x e^{x^2+y^2}] = e^{x^2+y^2}[1 + 2x^2].$$

$$\frac{\partial w}{\partial y} = xe^{x^2+y^2}[2y]$$

(e) $w = \cos(ye^{xy})\sin x$.

$$\frac{\partial w}{\partial x} = -\sin(ye^{xy}) \cdot y^2 e^{xy} \sin x + \cos(ye^{xy})\cos x.$$

$$\frac{\partial w}{\partial y} = -\sin(ye^{xy}) \cdot [e^{xy} + yxe^{xy}] \sin x.$$

5. Find the equation of the plane tangent to the surface $z = x^2 + y^3$ at $(3,1,10)$.

Recall: If $F:\mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\vec{x}_0 = (x_0, y_0)$ then the tangent plane of the graph of F at the point $(x_0, y_0, F(x_0,y_0))$ in $\mathbb{R}^3$ is defined by the eq^n:

$$z = F(x_0,y_0) + \left[\frac{\partial F}{\partial x}(x_0,y_0)\right](x-x_0) + \left[\frac{\partial F}{\partial y}(x_0,y_0)\right](y-y_0).$$

$\frac{\partial F}{\partial x}(3,1) = 2x\big|_{(3,1)} = 6.$ $\quad \frac{\partial F}{\partial y}(3,1) = 3y^2\big|_{(3,1)} = 3.$

$F(3,1) = 10.$

$$z = 10 + 6(x-3) + 3(y-1) = 6x + 3y - 11.$$

9.ⓒ Compute the matrix of partial derivatives of $F:\mathbb{R}^3 \to \mathbb{R}^2$, $F(x,y,z) = (\underbrace{x + e^z + y}_{F_1}, \underbrace{yx^2}_{F_2})$.

$$DF(x,y,z) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_3}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 & 1 & e^z \\ 2xy & x^2 & 0 \end{bmatrix}.$$

16. ⓒ Use the linear approximation to approximate a suitable function $\boxed{f(x,y)}$ & thereby estimate $\sqrt{(4.01)^2 + (3.98)^2 + (2.02)^2}$.

I think this is a typo... should say $f(x,y,z)$.

Recall: The linear approximation of a function f at (x_0, y_0) is given by the eqn of the tangent plane at the point $(x_0, y_0, f(x_0, y_0))$.

This generalizes for a function of 3 variables $f(x,y,z)$:

$$f(x_0,y_0,z_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0,z_0)\right](x-x_0) + \left[\frac{\partial f}{\partial y}(x_0,y_0,z_0)\right](y-y_0) + \left[\frac{\partial f}{\partial z}(x_0,y_0,z_0)\right](z-z_0).$$

Let $f(x,y,z) = \sqrt{x^2+y^2+z^2}$ & let's linearly approximate it at the point $(x_0,y_0,z_0) = (4,4,2)$:

$f(4,4,2) = \sqrt{16+16+4} = \sqrt{36} = 6.$

$\frac{\partial f}{\partial x}(4,4,2) = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2x = \frac{4}{6} = \frac{2}{3}.$

$\frac{\partial f}{\partial y}(4,4,2) = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2y = \frac{4}{6} = \frac{2}{3}.$

$\frac{\partial f}{\partial z}(4,4,2) = \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2z = \frac{2}{6} = \frac{1}{3}.$

So, we have $6 + \frac{2}{3}(x-4) + \frac{2}{3}(y-4) + \frac{1}{3}(z-2)$.

At the point $(4.01, 3.98, 2.02)$ this evaluates to:

$6 + \frac{2}{3}(0.01) + \frac{2}{3}(-0.02) + \frac{1}{3}(0.02)$

$= 6 + \frac{2}{300} - \frac{4}{300} + \frac{2}{300} = \boxed{6}.$

The actual value is $\sqrt{(4.01)^2 + (3.98)^2 + (2.02)^2} = 6.000749...$

19. (b) Compute the gradient of $f(x,y,z) = \frac{xyz}{x^2+y^2+z^2}$.

Recall: When we have $F: U \subset \mathbb{R}^n \to \mathbb{R}$, then by def^{n} $DF(x)$ is the $1 \times n$ matrix $\left[\frac{\partial f}{\partial x_1} \cdots \frac{\partial f}{\partial x_n}\right]$.
The gradient of F is
$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

$$\frac{\partial f}{\partial x} = \frac{yz[x^2+y^2+z^2]\,1 - xyz[2x]}{(x^2+y^2+z^2)^2} = \frac{yz[-x^2+y^2+z^2]}{(x^2+y^2+z^2)^2}.$$

$$\frac{\partial f}{\partial y} = \frac{xz[x^2+y^2+z^2] - xyz[2y]}{(x^2+y^2+z^2)^2} = \frac{xz[-y^2+x^2+z^2]}{(x^2+y^2+z^2)^2}.$$

$$\frac{\partial f}{\partial z} = \frac{xy(x^2+y^2+z^2) - xyz[2z]}{(x^2+y^2+z^2)^2} = \frac{xy[-z^2+x^2+y^2]}{(x^2+y^2+z^2)^2}.$$

$$\therefore \nabla f = \left(\frac{yz[-x^2+y^2+z^2]}{(x^2+y^2+z^2)^2}, \frac{xz[-y^2+x^2+z^2]}{(x^2+y^2+z^2)^2}, \frac{xy[-z^2+x^2+y^2]}{(x^2+y^2+z^2)^2}\right).$$

28. Suppose $F: \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. What is the derivative of F?

Recall: A map $F: \mathbb{R}^n \to \mathbb{R}^m$ is linear if
$\boxed{1}$ $F(x+y) = F(x) + F(y)$ & $\boxed{2}$ $F(kx) = kF(x)$
$\forall\, x, y \in \mathbb{R}^n$, $k \in \mathbb{R}$.

Recall: $F: U \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0 \in U$ if the partial derivatives of F exist at x_0 & if
$$\lim_{x \to x_0} \frac{\|F(x) - F(x_0) - DF(x_0)(x-x_0)\|}{\|x - x_0\|} = 0,$$
where $DF(x_0)$ is the $m \times n$ matrix $\left[\frac{\partial F_i}{\partial x_j}\right]\Big|_{x_0}$ is the derivative of F at x_0.

Suppose F is linear & $DF(x_0)$ is the derivative of F.

$$\Rightarrow \lim_{x \to x_0} \frac{\| F(x) - F(x_0) - DF(x_0)(x - x_0) \|}{\| x - x_0 \|} = 0$$

$$\Rightarrow \lim_{x \to x_0} \frac{\| F(x - x_0) - DF(x_0)(x - x_0) \|}{\| x - x_0 \|} = 0.$$

We want to find a $DF(x_0)$ to make this true. If we choose $DF(x_0) = F$ the numerator will vanish for all $(x - x_0)$ $\Rightarrow$ the derivative of F is F itself.

e.g.) let $n = 3$ & $m = 2$. $F: \mathbb{R}^3 \to \mathbb{R}^2$ is linear $\Rightarrow$ F is of the form:

$$F(x, y, z) = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}_{F} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z \\ a_{21}x + a_{22}y + a_{23}z \end{bmatrix}$$

$$= (\underbrace{a_{11}x + a_{12}y + a_{13}z}_{F_1}, \underbrace{a_{21}x + a_{22}y + a_{23}z}_{F_2}).$$

$$DF(x_0) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}\Bigg|_{x_0} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = F.$$

We can see that this generalizes for the $m \times n$ case.

- Consider $F(x,y) = (x^3+y^3)^{1/3}$. Find $\frac{\partial f}{\partial x}(0,0)$.

Note: $\frac{\partial f}{\partial x}(x,y) = \frac{1}{3}(x^3+y^3)^{-2/3}[3x^2] = \frac{3x^2}{3(x^3+y^3)^{2/3}}$.

At $(x,y) = (0,0)$ $\frac{\partial f}{\partial x}(x,y)$ is not defined. This does not necessarily mean $\frac{\partial f}{\partial x}(0,0)$ does not exist... it tells us we must go back to the formal definition of partial derivatives.

Recall: The partial derivative of $F: U \subset \mathbb{R}^2 \to \mathbb{R}$ with respect to x is:

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{F(x+h,y) - F(x,y)}{h} \text{ if the limit exists.}$$

Here we have:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \to 0} \frac{F(0+h,0) - F(0,0)}{h} = \lim_{h \to 0} \frac{(h^3+0^3)^{1/3} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1.$$

$$\therefore \boxed{\frac{\partial f}{\partial x}(0,0) = 1}.$$