6. Let $F(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) Compute the limit as $(x,y) \to (0,0)$ of F along the path $x = 0$.

$$\lim_{(0,y)\to(0,0)} F(x,y) = \lim_{(0,y)\to(0,0)} \frac{(0)y^3}{(0)^2+y^6} = \lim_{(0,y)\to(0,0)} \frac{0}{y^6} = 0.$$

(b) Compute the limit as $(x,y) \to (0,0)$ of F along the path $x = y^3$:

$$\lim_{(y^3,y)\to(0,0)} F(x,y) = \lim_{(y^3,y)\to(0,0)} \frac{y^3 y^3}{(y^3)^2+y^6} = \lim_{(y^3,y)\to(0,0)} \frac{y^6}{y^6+y^6}$$

$$= \lim_{(y^3,y)\to(0,0)} \frac{y^6}{2y^6} = \frac{1}{2}.$$

(c) Show that F is <u>not</u> continuous at $(0,0)$.

<u>Def^{n}</u>: $F: A \subset \mathbb{R}^n \to \mathbb{R}^m$ is <u>continuous</u> at x_0
$\iff \lim_{x \to x_0} F(x) = F(x_0)$.

We showed along the path $x = 0$ $\lim_{(x,y) \to (0,0)} F(x,y) = 0$

& along the path $x = y^3$ we have $\lim_{(x,y) \to (0,0)} F(x,y) = \frac{1}{2}$.

If $\lim_{(x,y) \to (0,0)}$ existed, then it must approach a value L as (x,y) gets near $(0,0)$ regardless of the path we take.

So, we just showed $\lim_{(x,y) \to (0,0)} F(x,y)$

does not exist $\Rightarrow$ F is <u>not</u> continuous at $(0,0)$.

9. Compute the following limits if they exist:

ⓐ $\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y} = \lim_{(x,y)\to(0,0)} x\left[\frac{e^{xy}-1}{xy}\right]$.

$= \left[\lim_{(x,y)\to(0,0)} x\right] \lim_{(x,y)\to(0,0)} \left[\frac{e^{xy}-1}{xy}\right]$ if both of those limits exist. [Theorem 3B (iii)].

$\lim_{(x,y)\to(0,0)} x = 0.$

$\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy} = \lim_{(x,y)\to(0,0)} f(g(x,y))$ where

$g: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto xy$, $f: \mathbb{R} \to \mathbb{R}$, $z \mapsto \frac{e^{z}-1}{z}$.

$\lim_{(x,y)\to(0,0)} xy = 0$ & $\lim_{z\to 0} \frac{e^{z}-1}{z} = \lim_{z\to 0} \frac{e^{z}}{1} = 1.$

L'Hôpital's Rule

So, $\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy} = \lim_{z\to 0} \frac{e^{z}-1}{z} = 1$.

$\therefore \lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{y} = \left[\lim_{(x,y)\to(0,0)} x\right]\left[\lim_{(x,y)\to(0,0)} \frac{e^{xy}-1}{xy}\right]$

$= [0][1] = 0.$

• Theorem 5 [Continuity of Compositions]: Let $g: A \subseteq \mathbb{R}^n \to \mathbb{R}^m$ & $f: B \subseteq \mathbb{R}^m \to \mathbb{R}^p$ & suppose $g(A) \subset B$. If g is continuous at $x_0 \in A$ & f continuous at $g(x_0) \Rightarrow f\circ g$ continuous at x_0.

- [And of course $f\circ g$ continuous at $x_0 \Rightarrow$ the limit exists at x_0 and the limit is $f\circ g(x_0)$].

(b) $$\lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{x^2y^2} = \lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{(xy)^2}.$$

Here I'm again going to use this composition of functions argument:

$$g: \mathbb{R}^2 \to \mathbb{R} \qquad f: \mathbb{R}\to\mathbb{R}$$
$$(x,y) \to xy. \qquad z \to \frac{\cos z - 1}{z^2}.$$

$$\lim_{(x,y)\to 0} xy = \left[\lim_{(x,y)\to(0,0)} x\right]\left[\lim_{(x,y)\to(0,0)} y\right] = 0\cdot 0 = 0.$$

$$\lim_{z\to 0} \frac{\cos z - 1}{z^2} = \lim_{z\to 0} \frac{-\sin z}{2z} = \lim_{z\to 0} \frac{-\cos z}{2} = \frac{-1}{2}.$$

L'Hôpital's Rule

$$\therefore \lim_{(x,y)\to(0,0)} \frac{\cos(xy)-1}{(xy)^2} = \frac{-1}{2}.$$

17. Find $\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\log(x^2+y^2)$.
[Hint: use polar coordinates.]

$$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\} \quad x^2+y^2 = r^2$$

$$\lim_{(x,y)\to(0,0)} (3x^2+3y^2)\log(x^2+y^2)$$

$$= \lim_{r\to 0} \left[3r^2(\cos^2\theta + \sin^2\theta)\right] \log\left(r^2(\cos^2\theta+\sin^2\theta)\right)$$

$$= \lim_{r\to 0} 3r^2 \log(r^2) = 3\lim_{r\to 0} \frac{\log(r^2)}{r^{-2}} = 3\lim_{r\to 0} \frac{\frac{1}{r^2}\cdot 2r}{-2r^{-3}}$$

$$= 3\lim_{r\to 0} \frac{2/r}{-2/r^3} = 3\lim_{r\to 0} \frac{2}{r}\cdot\frac{r^3}{-2} = 3\lim_{r\to 0} -r^2 = 0.$$

25. ⓐ Can $\frac{\sin(x+y)}{x+y}$ be made continuous by suitably defining it at $(0,0)$?

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x+y)}{x+y} = \lim_{t\to 0} \frac{\sin t}{t} = \lim_{t\to 0} \frac{\cos t}{1} = 1.$$

Yes, if we define:

$$F(x,y) = \begin{cases} \frac{\sin(x+y)}{x+y} & \text{if } (x,y) \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

then $F(x,y)$ is continuous.

35. (a) Find a specific number $\delta > 0$ s.t. if $|a| < \delta$, then $|a^3 + 3a^2 + a| < 1/100$.

For now let's just suppose $0 < \delta < 1$. Then if $|a| < \delta < 1 \Rightarrow |a| < 1 \Rightarrow a^2 < |a|$ & $a^3 < |a|$.

Then $|a^3 + 3a^2 + a| \leq |a^3| + |3a^2| + |a|$

Triangle inequality $|x+y| \leq |x| + |y|$.

$= |a^3| + 3|a^2| + |a| < |a| + 3|a| + |a| = 5|a| < 1/100$

$\Leftrightarrow \boxed{|a| < 1/500}$. So. we should choose $\delta = 1/500$.