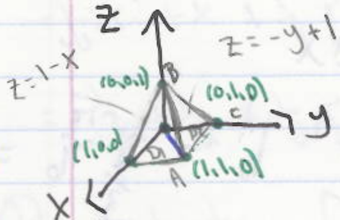
$\boxed{21.}$ Evaluate $\iiint_W (1-z^2)\,dx\,dy\,dz$; W is the pyramid with top vertex at $(0,0,1)$ & base vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, & $(1,1,0)$.



$1-x=-y+1$

$\Rightarrow x=y.$

$\vec{AB}=(1,1,-1)$. $\vec{CB}=(0,1,-1)$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (0,1,1).$$

$0(x-0)+(y-0)+(z-1)=0$

$y+z-1=0$

$z=-y+1.$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix}$$

$=(1,0,1).$

$x+(z-1)=0$

$z=1-x$

$$\iint_{D_1}\int_0^{1-x}(1-z^2)\,dz\,dy\,dx+\iint_{D_2}\int_0^{-y+1}(1-z^2)\,dz\,dx\,dy$$

$u=-y+1$

$du=-dy$

$$=\int_0^1\int_0^x z-\tfrac{1}{3}z^3\Big|_0^{1-x}dy\,dx+\int_0^1\int_0^y z-\tfrac{1}{3}z^3\Big|_0^{-y+1}dx\,dy$$

$$=\int_0^1\int_0^x 1-x-\tfrac{1}{3}(1-x)^3\,dy\,dx+\int_0^1\int_0^y -y+1-\tfrac{1}{3}(-y+1)^3\,dx\,dy$$

$(1-x)(1-x)^2$

$(1-2x+x^2)(1-x)$

$= 1+2x^2-x-2x+x^2-x^3$

$= -x^3+3x^2-3x+1$

$$= \int_0^1 x-x^2-\frac{1}{3}(-x^4+3x^3-3x^2+x)\,dx + \int_0^1 -y^2+y-\frac{1}{3}(-y^4+3y^3-3y^2+y$$

$$= \frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{3}\left[-\frac{1}{5}x^5+\frac{3}{4}x^4-x^3+\frac{1}{2}x^2\right]\Big|_0^1$$

$$+\left[-\frac{1}{3}y^3+\frac{1}{2}y^2-\frac{1}{3}\left[-\frac{1}{5}y^5+\frac{3}{4}y^4-y^3+\frac{1}{2}y^2\right)\Big|_0^1\right]$$

$$= \frac{1}{2}-\frac{1}{3}-\frac{1}{3}\left[-\frac{1}{5}+\frac{3}{4}-1+\frac{1}{2}\right]$$

$$+ -\frac{1}{3}+\frac{1}{2}-\frac{1}{3}\left[-\frac{1}{5}+\frac{3}{4}-1+\frac{1}{2}\right]$$

$$= \frac{1}{3}-\frac{2}{3}\left[-\frac{1}{5}+\frac{1}{4}\right] = \frac{1}{3}-\frac{2}{3}\left[\frac{-4}{20}+\frac{5}{20}\right]$$

$$= \frac{1}{3}-\frac{2}{3}\cdot\frac{1}{20} = \frac{10}{30}-\frac{1}{30} = \frac{9}{30} = \boxed{\frac{3}{10}}.$$

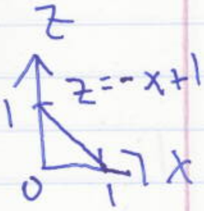
Perhaps it would have been easier to just solve:

$$\int_0^1 \int_0^{-x+1} \int_0^{1-z} (1-z^2)\; dydzdx$$

$= \ldots = \frac{3}{10}.$

$y = 1-z$

21. $\int_0^1 \int_0^{-x+1} \int_0^{1-z} (1-z^2)\,dy\,dz\,dx$



$= \int_0^1 \int_0^{-x+1} (1-z^2)(1-z)\,dz\,dx$

$= \int_0^1 \int_0^{-x+1} 1-z-z^2+z^3\,dz\,dx$

$= \int_0^1 z-\frac{1}{2}z^2-\frac{1}{3}z^3+\frac{1}{4}z^4 \Big|_0^{-x+1}$

$= \int_0^1 (-x+1)-\frac{1}{2}(-x+1)^2-\frac{1}{3}(-x+1)^3+\frac{1}{4}(-x+1)^4$

$= -\int_0^{*} u-\frac{1}{2}u^2-\frac{1}{3}u^3+\frac{1}{4}u^4\,du$

$= -\left[\frac{1}{2}(-x+1)^2-\frac{1}{6}(-x+1)^3-\frac{1}{12}(-x+1)^4+\frac{1}{20}(-x+1)^5\right]\Big|$

$= +\underbrace{\frac{1}{2}-\frac{1}{6}}_{\frac{1}{3}} -\frac{1}{12}+\frac{1}{20}$

$\underbrace{\frac{1}{3}-\frac{1}{12}}$

$\underbrace{\frac{4}{12}-\frac{1}{12}}$

$\underbrace{\frac{3}{12}}$

$\underbrace{\frac{1}{4}}_{\frac{5}{20}}$

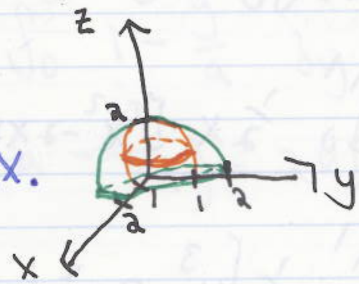
$\frac{6}{20} = \boxed{\frac{3}{10}}$

27. $W = \{(x,y,z) \mid x^2+y^2 \le 1,\ z \ge 0 \ \& \ x^2+y^2+z^2 \le 4\}$.

$0 \le z^2 \le 4 - x^2 - y^2$
$\Rightarrow 0 \le z \le \sqrt{4-x^2-y^2}$.

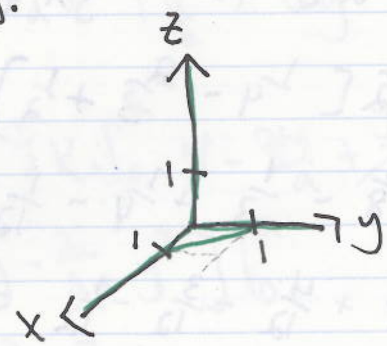
$0 \le x^2+y^2 \le 1 \Rightarrow 3 \le 4 - \underbrace{(x^2+y^2)}_{\le 1} \le 4$.

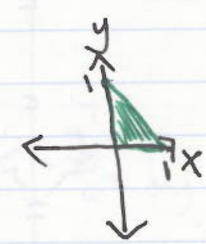
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx.$$



30. Let W be the region bounded by the planes $x=0$, $y=0$, $z=0$, $x+y=1$, & $z=x+y$.

ⓐ Find the volume of W.





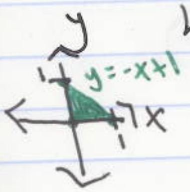
$x+y-1 = x+y-z \Rightarrow z=1$.

$$\int_0^1\int_0^{-x+1}\int_0^{x+y} dz\,dy\,dx$$

$= \int_0^1\int_0^{-x+1} x+y \; dy\,dx = \int_0^1 xy + \frac{1}{2}y^2 \Big|_0^{-x+1} dx = \int_0^1 x(-x+1) + \frac{1}{2}(-x+1)^2 dx$

$= \int_0^1 -x^2 + x + \frac{1}{2}[x^2 - 2x + 1]\,dx = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{2}\left[\frac{1}{3}x^3 - x^2 + x\right]\Big|_0^1$

$= -\frac{1}{3} + \frac{1}{2} + \frac{1}{2}\left[\frac{1}{3} - 1 + 1\right] = -\frac{1}{3} + \frac{1}{2} + \frac{1}{6} = -\frac{2}{6} + \frac{3}{6} + \frac{1}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$.



4. Find the average of $f(x,y) = e^{x+y}$ over the triangle with vertices $(0,0)$, $(0,1)$, & $(1,0)$.

Recall: The average value of f over D is:

$$[f]_{av} = \frac{\iint_D f(x,y)\,dxdy}{\iint_D dxdy}.$$

$$\iint_D dxdy = \int_0^1 \int_0^{-x+1} dydx = \int_0^1 -x+1\,dx = -\frac{x^2}{2} + x \Big|_0^1$$

$$= -\frac{1}{2} + 1 = \boxed{\frac{1}{2}}.$$

$u = x+y$
$du = dy$

$$\iint_D f(x,y)\,dxdy = \int_0^1 \int_0^{-x+1} e^{x+y}\,dy\,dx$$

$$= \int_0^1 e^{x+y} \Big|_0^{-x+1} dx = \int_0^1 e^1 - e^x\,dx$$

$$= e^1 x - e^x \Big|_0^1 = e^1 - e^1 - [0-1] = 1.$$

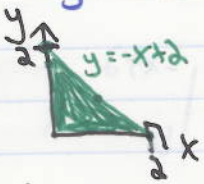
So, $[f]_{av} = \frac{1}{1/2} = \boxed{2}.$

$z = 2-x-y.$

If $z=0$:

$x+y=2$

$\Rightarrow y = 2-x.$



13. Find the center of mass of the region bounded by $x+y+z=2$, $x=0$, $y=0$, & $z=0$, assuming the density to be uniform.

- Uniform density $\Rightarrow \delta(x,y) = K$, for some constant K.
- The mass is given by: $\int_0^2 \int_0^{-x+2} \int_0^{2-x-y} K \, dz\,dy\,dx$

$$= K\int_0^2 \int_0^{-x+2} 2-x-y \; dy\,dx = K\int_0^2 2y - xy - \tfrac{1}{2}y^2 \Big|_0^{-x+2} dx$$

$$= K\int_0^2 (2-x)(-x+2) - \tfrac{1}{2}(x^2-4x+4)\; dx$$

$$= K\int_0^2 -2x+4+x^2-2x-\tfrac{1}{2}x^2+2x-2 \; dx$$

$$\bar{x}_i = \frac{\iiint_W x_i\,\delta(x,y,z)\,dx\,dy\,dz}{\underbrace{\iiint_W \delta(x,y,z)\,dx\,dy\,dz}_{\text{mass}}}$$

$$= K\int_0^2 \tfrac{1}{2}x^2 - 2x + 2\; dx = K\left[\tfrac{1}{6}x^3 - x^2 + 2x\right]_0^2$$

$= K\left[\frac{8}{6} - 4 + 4\right] = \frac{4}{3}K.$

• $K\int_0^2\int_0^{-x+2}\int_0^{2-x-y} x \, dz\,dy\,dx = K\int_0^2\int_0^{-x+2} 2x - x^2 - xy \, dy\,dx$

$= K\int_0^2 2xy - x^2y - \frac{1}{2}xy^2 \Big|_0^{-x+2} dx$

$= K\int_0^2 -2x^2 + 4x + x^3 - 2x^2 - \frac{1}{2}x[x^2 - 4x + 4] \, dx$

$= K\int_0^2 x^3 - 4x^2 + 4x - \frac{1}{2}x^3 + 2x^2 - 2x \, dx$

$= K\int_0^2 \frac{1}{2}x^3 - 2x^2 + 2x \, dx = K\left[\frac{1}{8}x^4 - \frac{2}{3}x^3 + x^2\right]\Big|_0^2$

$= K\left[2 - \frac{16}{3} + 4\right] = K\left[\frac{18}{3} - \frac{16}{3}\right] = \frac{2}{3}K$

• $K\int_0^2\int_0^{-x+2} 2y - xy - y^2 \, dy\,dx = K\int_0^2 y^2 - \frac{1}{2}xy^2 - \frac{1}{3}y^3\Big|_0^{-x+2} dx$

$= K\int_0^2 x^2 - 4x + 4 - \frac{1}{2}[x^3 - 4x^2 + 4x] - \frac{1}{3}[-x^3 + 2x^2 + 4x^2 - 8x - 4x + 8] \, dx$

$= K\int_0^2 x^2 - 4x + 4 - \frac{1}{2}x^3 + 2x^2 - 2x + \frac{1}{3}x^3 - \frac{2}{3}x^2 - \frac{4}{3}x^2 + \frac{8}{3}x + \frac{4}{3}x - \frac{8}{3} \, dx$

$-\frac{1}{2} + \frac{1}{3} = -\frac{3}{6} + \frac{2}{6} = -\frac{1}{6}$ $= K\int_0^2 -\frac{1}{6}x^3 + x^2 - 2x + \frac{4}{3} \, dx = K\left[-\frac{1}{24}x^4 + \frac{1}{3}x^3 - x^2 + \frac{4}{3}x\right]\Big|_0^2$

$= K\left[-\frac{16}{24} + \frac{8}{3} - 4 + \frac{8}{3}\right] = K\left[-\frac{2}{3} + \frac{16}{3} - \frac{12}{3}\right] = \frac{2}{3}K.$

$u = 2-x-y$, $du = -dy$ • $K\int_0^2\int_0^{-x+2} \frac{1}{2}(2-x-y)^2 \, dy\,dx = -K\int_0^2 \frac{1}{6}(2-x-y)^3\Big|_0^{-x+2} dx$

$= -K\int_0^2 \frac{1}{6}(\underbrace{2-\cancel{x}+x-2}_{0})^3 - \frac{1}{6}(2-x)^3 \, dx$

$u = 2-x$, $du = -dx$ $= \frac{K}{6}\int_0^2 (2-x)^3 \, dx = -\frac{K}{6}\cdot\frac{1}{4}(2-x)^4\Big|_0^2$

$= 0 + \frac{K}{24}2^4 = \frac{16}{24}K = \frac{2}{3}K.$

• So, the center of mass is: $(\bar{x}, \bar{y}, \bar{z}) = \frac{3}{4K}\left(\frac{2}{3}K, \frac{2}{3}K, \frac{2}{3}K\right) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$

$$\frac{|2x\,x^2+(-y\,y^2)|}{|x^2+y^2|} \le \frac{|2x[x^2+y^2]|}{x^2+y^2} + \frac{|-y\,y^2|}{x^2+y^2}$$

$$\frac{|2x[x^2+y^2]|}{x^2+y^2} = 0 \qquad \frac{|-y\,y^2|}{x^2+y^2} \le \frac{y|x^2+y^2|}{(x^2+y^2)} = 0$$

Question 4.

(a) Evaluate $\lim_{(x,y)\to(0,0)} \frac{2x^3-y^3}{x^2+y^2}$. Remember to justify all steps leading to your final answer. [8 marks]

• If this limit exists we're expecting it to be "0", since

$$\lim_{(0,y)\to(0,0)} = \lim_{y\to 0}\frac{-y^3}{y^2} = \lim_{y\to 0} -y = 0.$$

$|x^2+y^2|(x-y)$

$$0 \le \lim_{(x,y)\to(0,0)} \frac{|2x^3+(-y^3)|}{|x^2+y^2|} \underset{\text{Δ-inequality}}{\le} \lim_{(x,y)\to(0,0)} \frac{|2x^3|}{|x^2+y^2|} + \frac{|y^3|}{|x^2+y^2|}$$

$$\le \lim_{(x,y)\to(0,0)} \frac{2|x^2|\,|x|}{|x^2+y^2|} + \frac{|y^2|\,|y|}{|x^2+y^2|} \le \lim_{(x,y)\to(0,0)} 2\frac{|x^2+y^2|\,|x|}{|x^2+y^2|} + \frac{|x^2+y^2|\,|y|}{|x^2+y^2|}$$

$x^2 \le x^2+y^2$ & $y^2 \le x^2+y^2$

$$\le \lim_{(x,y)\to(0,0)} 2|x| + |y| = 0 \quad \text{since} \quad \lim_{(x,y)\to(0,0)} 2|x| = 0 \ \& \ \lim_{(x,y)\to(0,0)} |y| =$$

$$\therefore \text{By the Squeeze Theorem, } \lim_{(x,y)\to(0,0)} \frac{2x^3-y^3}{x^2+y^2} = 0.$$

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NAME:

Question 4 (cont'd)

(b) Let $\vec{u} = (a, b)$ be a unit vector. Using the definition of the directional derivative, find the choices of (a, b) for which the directional derivative of the function $f(x, y) = x^{\frac{1}{3}}y^{\frac{1}{3}}$ at $(0, 0)$ in the direction of $\vec{u}$ exists. Comment **briefly** on the significance of your findings. [9 marks]

We need to use the formal def^{in} here since $\nabla f(x,y) = \left(\frac{1}{3}x^{-\frac{2}{3}}y^{\frac{1}{3}}, \frac{1}{3}x^{\frac{1}{3}}y^{-\frac{2}{3}}\right)$ not defined at $(0,0)$.

$$Df_{(0,0)}(a,b) = \frac{d}{dt}\Big|_{t=0} \underbrace{f((0,0) + t(a,b))}_{g(t)} = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{f((0,0) + h(a,b)) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{a^{\frac{1}{3}}h^{\frac{1}{3}}h^{\frac{1}{3}}b^{\frac{1}{3}}}{h} = \lim_{h\to 0} \frac{a^{\frac{1}{3}}b^{\frac{1}{3}}}{h^{\frac{1}{3}}}.$$

If $a \neq 0$ & $b \neq 0$ then this limit is ∞.

If $a = 0$ or $b = 0 \Rightarrow$ this limit is 0. So, the directional derivative of $f(x,y)$ exists for unit vectors of the form $\vec{u} = (0,b)$ or $\vec{u} = (a,0)$. i.e. vectors of the form $(\pm 1, 0)$ & $(0, \pm 1)$. We would have $Df_{(0,0)}(a,0) = Df_{(0,0)}(0,b) = 0$.

So, this this function is s.t. its partial derivatives exist [direction $(\pm 1, 0)$ is $\frac{\partial f}{\partial x}$ & $(0, \pm 1)$ is $\frac{\partial f}{\partial y}$], but it has no other directional derivatives.

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