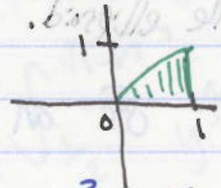
$\boxed{5.}$ Change the order of integration & evaluate

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dx\, dy$$



$x = \sqrt{y}$
$\Rightarrow y = x^2.$

$$= \int_0^1 \int_0^{x^2} e^{x^3}\, dy\, dx = \int_0^1 y e^{x^3} \Big|_0^{x^2} dx$$

$u = x^3$
$du = 3x^2\, dx$
$\frac{1}{3} du = x^2 dx$

$$= \int_0^1 x^2 e^{x^3}\, dx = \frac{1}{3}\int e^u\, du = \frac{1}{3} e^{x^3}\Big|_0^1 = \frac{1}{3}[e^1 - 1].$$

$\boxed{9.}$ If $D = [-1,1] \times [-1,2]$, show that $1 \le \iint_D \frac{dx\,dy}{x^2+y^2+1} \le 6.$

<u>Mean-Value Inequality:</u> Suppose there are numbers m & M s.t. $\forall (x,y) \in D$ & $m \le f(x,y) \le M$.

Then, integrating over D we have $m \cdot A(D) \le \iint_D f(x,y)\, dA \le M \cdot A(D)$ where $A(D)$ is the area of the region D.

Here $A(D) = 2 \cdot 3 = 6.$

$$x^2+y^2 \ge 0 \Rightarrow x^2+y^2+1 \ge 1 \Rightarrow \frac{1}{x^2+y^2+1} \le 1.$$

On $\overset{\text{lo a c}}{D}$ we have $f(0,0) = \frac{1}{1} = 1.$

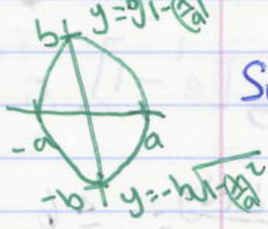
The largest pair we can put in x^2+y^2+1 would be $(1,2)$. $f(1,2) = \frac{1}{1+4+1} = \frac{1}{6}$. So, $\frac{1}{6} \le \frac{1}{x^2+y^2+1}$.

$$\therefore \frac{1}{6} \le \frac{1}{x^2+y^2+1} \le 1 \Rightarrow 1 \le \iint_D \frac{dx\,dy}{x^2+y^2+1} \le 6$$

by the Mean-Value Inequality.

11. Compute the volume of the ellipsoid with semiaxes a, b, & c.
Hint: Use symmetry & first find the volume of one half of the ellipsoid.

$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$



The eq'n of an ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Solving for z: $z = \pm c\underbrace{\sqrt{1-\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}}_{\leq 1}$.

So, we want to integrate $2\int_{-a}^{a}\int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}} c\sqrt{1-\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}\, dy\, dx$

$$= 4\int_{-a}^{a}\int_{0}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}} c\sqrt{1-\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2}\, dy\, dx$$

$$= 4c\int_{-a}^{a}\int_{0}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}} \sqrt{1-\left(\frac{x}{a}\right)^2}\sqrt{1-\frac{y^2}{b^2\left(1-\left(\frac{x}{a}\right)^2\right)}}\, dy\, dx$$

$y = b\sqrt{1-\left(\frac{x}{a}\right)^2}\sin\theta$

$dy = b\sqrt{1-\left(\frac{x}{a}\right)^2}\cos\theta$

$y = b\sqrt{1-\left(\frac{x}{a}\right)^2}$

$\Rightarrow \sin\theta = 1$

$\Rightarrow \theta = \frac{\pi}{2}$

$\Rightarrow \cos\theta = 0$.

$y = 0 \Rightarrow \sin\theta = 0$

$\Rightarrow \theta = 0$

$\Rightarrow \cos\theta = 1$.

$$= 4c\int_{-a}^{a}\int_{0}^{\frac{\pi}{2}} \sqrt{1-\left(\frac{x}{a}\right)^2}\sqrt{1-\sin^2\theta}\; b\sqrt{1-\left(\frac{x}{a}\right)^2}\cos\theta\, d\theta\, dx$$

$$= 4cb\int_{-a}^{a}\int_{0}^{\frac{\pi}{2}}\left[1-\left(\frac{x}{a}\right)^2\right]\cos^2\theta\, d\theta\, dx$$

$$= 4bc\int_{-a}^{a}\int_{0}^{\frac{\pi}{2}}\left[1-\left(\frac{x}{a}\right)^2\right]\left[\frac{1}{2}\cos(2\theta)+\frac{1}{2}\right] d\theta\, dx$$

$$= 4bc\int_{-a}^{a}\left[1-\left(\frac{x}{a}\right)^2\right]\left[\frac{1}{4}\sin(2\theta)+\frac{1}{2}\theta\right]\Big|_0^{\frac{\pi}{2}} dx$$

$$= 4bc\int_{-a}^{a}\left[1-\left(\frac{x}{a}\right)^2\right]\left[0+\frac{\pi}{4}-0\right] dx$$

$$= bc\pi\int_{-a}^{a} 1-\frac{x^2}{a^2}\, dx = bc\pi\left[x-\frac{x^3}{3a^2}\right]\Big|_{-a}^{a}$$

$\cos^2\theta = \frac{\cos(2\theta)+1}{2}$

$$= bc\pi\left[a-\frac{1}{3}a+a+\frac{-a}{3}\right] = bc\pi\left[2a-\frac{2}{3}a\right]$$

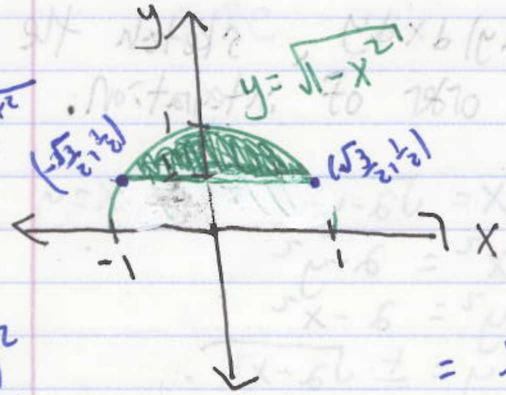
$\sin(2\theta) = 2\sin\theta\cos\theta$

$$= \boxed{\frac{4}{3}abc\pi}.$$

Note: Same as $2\int_{-a}^{a}\int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}}\int_{0}^{c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}} dz\,dy\,dx$!

15. Evaluate $\iint_D y^3 (x^2+y^2)^{-3/2}\, dx\,dy$, where D is the region determined by the conditions $\frac{1}{2} \le y \le 1$ & $x^2+y^2 \le 1$.

$y = \frac{1}{2}$
$\Rightarrow \frac{1}{2} = \sqrt{1-x^2}$
$\Rightarrow -\frac{1}{4}+1 = x^2$
$\Rightarrow x = \pm \frac{\sqrt{3}}{2}$



$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int_{\frac{1}{2}}^{\sqrt{1-x^2}} y^3 (x^2+y^2)^{-3/2}\, dy\, dx$$

$$= \frac{1}{2} \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int [u-x^2]\, u^{-3/2}\, du\, dx$$

$u = x^2+y^2$
$du = 2y\,dy$
$\frac{1}{2} du = y\,dy$
$y^2 = u - x^2$

$$= \frac{1}{2} \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \int u^{-\frac{1}{2}} - x^2 u^{-3/2}\, du\, dx$$

$$= \frac{1}{2} \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} 2u^{\frac{1}{2}} + 2x^2 u^{-\frac{1}{2}} \Big|\, dx$$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \sqrt{x^2+y^2} + \frac{x^2}{\sqrt{x^2+y^2}} \Big|\, dx = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{2x^2+y^2}{\sqrt{x^2+y^2}} \Big|_{\frac{1}{2}}^{\sqrt{1-x^2}} dx$$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{2x^2+1-x^2}{\sqrt{\cancel{x^2}+1-\cancel{x^2}}} - \frac{2x^2+\frac{1}{4}}{\sqrt{x^2+\frac{1}{4}}}\, dx = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} x^2+1 - \frac{2x^2+\frac{1}{4}}{\sqrt{x^2+\frac{1}{4}}}\, dx$$

2, $\sqrt{3}$, 1, $\frac{\pi}{3}$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} x^2+1 - \frac{2(x^2+\frac{1}{8})}{\frac{1}{2}\sqrt{4x^2+1}}\, dx = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} x^2+1 - \frac{4x^2+1-\frac{1}{2}}{\sqrt{4x^2+1}}$$

$2x = \tan\theta$
$2dx = \sec^2\theta\, d\theta$
$dx = \frac{1}{2}\sec^2\theta\, d\theta$
$\tan\theta = 2\cdot\frac{\sqrt{3}}{2}$
$\Rightarrow \tan\theta = \sqrt{3}$
$\Rightarrow \theta = \frac{\pi}{3}$.
$\tan\theta = -\sqrt{3}$
$\Rightarrow \theta = -\frac{\pi}{3}$.

$$= \frac{1}{3}x^3 + x \Big|_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} - \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\tan^2\theta+1-\frac{1}{2}}{\sqrt{\tan^2\theta+1}}\, d\theta\ \sec^2\theta$$

$$\frac{1}{3}x^3 + x \Big|_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} - \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{\sec^2\theta - \frac{1}{2}}{\sec\theta}\, \sec^2\theta\, d\theta$$

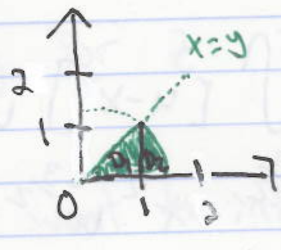
$$= \frac{1}{3}\frac{3\sqrt{3}}{8} + \frac{\sqrt{3}}{2} + \left[\frac{+\sqrt{3}}{8} + \frac{\sqrt{3}}{2}\right] - \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\sec^2\theta - \frac{1}{2}\right] \sec\theta\, d\theta$$

$\frac{d}{dx}\frac{1}{2}\tan\theta\sec\theta$
$= \frac{1}{2}\sec^3\theta + \frac{1}{2}\sec\theta\tan^2\theta$
$= \frac{1}{2}\sec^3\theta + \frac{1}{2}\sec\theta[-1+\sec^2\theta]$
$= \sec^3\theta - \frac{1}{2}\sec\theta$.

$$\boxed{\frac{\sqrt{3}}{4}}$$

$$= \frac{2\sqrt{3}}{8} + \sqrt{3} - \frac{1}{2}\left[\frac{1}{2}\tan\theta\sec\theta\right]\Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{5\sqrt{3}}{4} - \frac{1}{4}(\sqrt{3}\cdot 2 + [+\sqrt{3}\cdot 2]) = \frac{5\sqrt{3}}{4} - \frac{4\sqrt{3}}{4}$$

17. Given that the double integral $\iint_D f(x,y)\,dx\,dy$ of a positive continuous function f equals the iterated integral $\int_0^1 \int_y^{\sqrt{2-y^2}} f(x,y)\,dx\,dy$, sketch the region D & interchange the order of integration.



$x = \sqrt{2-y^2}$ $\qquad$ $x=y$

$\Rightarrow x^2 = 2-y^2$

$\Rightarrow y^2 = 2-x^2$

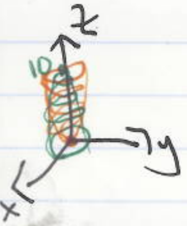
$\Rightarrow y = \pm\sqrt{2-x^2}$.

So, $\int_0^1 \int_y^{\sqrt{2-y^2}} f(x,y)\,dx\,dy = \int_0^1 \int_0^x f(x,y)\,dy\,dx$

$+ \int_1^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} f(x,y)\,dy\,dx.$

5.5:

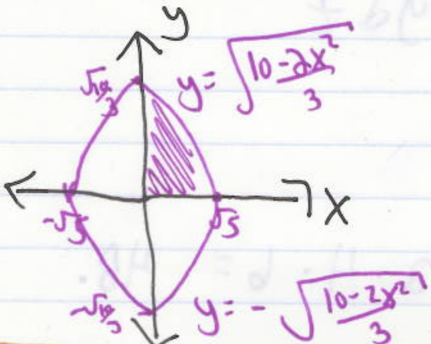
11. Find the volume of the region bounded by $z = x^2 + y^2$ & $z = 10 - x^2 - 2y^2$.



Need to know when $x^2 + y^2 = 10 - x^2 - 2y^2$

$\Rightarrow 2x^2 + 3y^2 = 10$. So, these 2 surfaces intersect

at the ellipse given by $2x^2 + 3y^2 = 10$.



$3y^2 = 10 - 2x^2$

$y^2 = \frac{10 - 2x^2}{3}$

$y = \pm\sqrt{\frac{10-2x^2}{3}}$

$$\boxed{\int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{\frac{10-2x^2}{3}}}^{\sqrt{\frac{10-2x^2}{3}}} \int_{x^2+y^2}^{10-x^2-2y^2} dz\,dy\,dx}$$

6.3: 15. Find the average value of $\sin^2 \pi z \cdot \cos^2 \pi x$ over the cube $[0,2] \times [0,4] \times [0,6]$.

Recall: The average value of a function f on a region W in 3-space is defined by

$$[f]_{av} = \frac{\iiint_W f(x,y,z)\,dx\,dy\,dz}{\iiint_W dx\,dy\,dz}.$$

Here we have:

$$\int_0^6 \int_0^4 \int_0^2 dx\,dy\,dz = 2 \cdot 4 \cdot 6 = 48.$$

$$\int_0^6 \int_0^4 \int_0^2 \sin^2(\pi z)\cos^2(\pi x)\,dx\,dy\,dz$$

$\cos^2 x = \frac{\cos(2x)+1}{2}$

$$= \frac{1}{2}\int_0^6 \int_0^4 \int_0^2 \sin^2(\pi z)\,[\cos(2\pi x)+1]\,dx\,dy\,dz$$

$\sin^2 x = \frac{1-\cos(2x)}{2}$

$$= \frac{1}{2}\int_0^6 \int_0^4 \sin^2(\pi z) \cdot \left[\frac{1}{2\pi}\sin(2\pi x) + x\right]\Big|_0^2 dy\,dz$$

$$= \frac{1}{2}\int_0^6 \sin^2(\pi z)\left[\frac{1}{2\pi}\cancel{\sin(4\pi)}^{\,0} + 2\right] y\,\Big|_0^4 dz$$

$$= \frac{1}{4}\int_0^6 [1-\cos(2\pi z)]\,8\,dz$$

$$= 2\left[z - \sin(2\pi z)\cdot\frac{1}{2\pi}\right]\Big|_0^6 = 2[6] = 12.$$

So, $[f]_{av} = \frac{12}{48} = \boxed{\frac{1}{4}}$.