13. Let (x_0, y_0, z_0) be a point of the locus defined by $z^2 + xy - a = 0$, $z^2 + x^2 - y^2 - b = 0$, where a & b are constants.

ⓐ Under what conditions may the part of the locus near (x_0, y_0, z_0) be represented in the form $x = f(z)$, $y = g(z)$?

Let $F_1(x,y,z) = z^2 + xy - a$, $F_2(x,y,z) = z^2 + x^2 - y^2 - b$.

$$\Delta = \left.\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix}\right|_{(x_0,y_0,z_0)} = \left.\begin{vmatrix} y & x \\ 2x & -2y \end{vmatrix}\right|_{(x_0,y_0,z_0)} = -2y_0^2 - 2x_0^2$$

$= 0 \iff x_0^2 + y_0^2 = 0 \iff (x_0, y_0) = (0,0)$.

So, part of the locus near (x_0, y_0, z_0) may be represented as $x = f(z)$ & $y = g(z)$ so long as $(x_0, y_0) \neq (0,0)$.

ⓑ Compute $f'(z)$ & $g'(z)$.

$\frac{\partial x}{\partial z} = x' = f'(z)$ can be ~~solved~~ found using implicit differentiation:

$2z + x'y + xy' = 0 \quad + \quad 2z + 2xx' - 2yy' = 0.$ ①

$*\frac{1}{2}x \qquad xz + x^2x' - xyy' = 0$

$*y \qquad \underline{2yz + y^2x' + xyy' = 0}$

$$2yz + xz + (x^2+y^2)x' = 0 \Rightarrow x' = \frac{-z(2y+x)}{x^2+y^2} = f'(z).$$

$*-\frac{1}{2}y \qquad -yz - xyx' + y^2y' = 0$

$*x \qquad \underline{2xz + xyx' + x^2y' = 0}$

$$2xz - yz + (x^2+y^2)y' = 0 \Rightarrow y' = \frac{z(-2x+y)}{x^2+y^2} = g'(z).$$

15. Find the volume bounded by the graph of $f(x,y) = x^4 + y^2$ & the rectangle $[-1,1] \times [-3,-2]$.

$$\int_{-3}^{-2}\int_{-1}^{1}\left(x^4+y^2\right)dx\,dy = \int_{-3}^{-2} \tfrac{1}{5}x^5 + y^2x\Big|_{-1}^{1}\,dy$$

$$= \int_{-3}^{-2}\left[\tfrac{1}{5}+y^2-\left[-\tfrac{1}{5}-y^2\right]\right]dy = \int_{-3}^{-2}\left(\tfrac{2}{5}+2y^2\right)dy$$

$$= \tfrac{2}{5}y+\tfrac{2}{3}y^3\Big|_{-3}^{-2} = \tfrac{2}{5}(-2)+\tfrac{2}{3}(-8)-\left[\tfrac{2}{5}(-3)+\tfrac{2}{3}(-27)\right]$$

$$= -\tfrac{4}{5}-\tfrac{16}{3}+\tfrac{6}{5}+18 = \tfrac{2}{5}-\tfrac{16}{3}+18 = \tfrac{6}{15}-\tfrac{80}{15}+\tfrac{270}{15} = \boxed{\tfrac{196}{15}}.$$

$$\begin{array}{r} 18 \\ \times 15 \\ \hline 90 \\ 180 \\ \hline 270 \end{array} \qquad \begin{array}{r} 276 \\ -\ 80 \\ \hline 196 \end{array}$$

$\boxed{5.}$ Sketch the solid whose volume is given by: $\int_0^1\int_0^1 (5-x-y)\,dy\,dx$

On the xy-axis we have $[0,1]\times[0,1]$.

$z = 5-x-y$. So, we want to sketch the part of the plane $z=5-x-y$ that falls above $[0,1]\times[0,1]$ & include everything b/w this graph & our rectangle.