7.6: Systems of Linear DE's

#1, 3, 9: Use the Laplace Transform to solve the given system of linear DE's.

1. $$\begin{cases} \frac{dx}{dt} = -x + y \\ \frac{dy}{dt} = 2x \end{cases}, \quad x(0) = 0,\ y(0) = 1.$$

$$\begin{cases} \mathcal{L}\{x'\} = -\mathcal{L}\{x\} + \mathcal{L}\{y\} \\ \mathcal{L}\{y'\} = 2\mathcal{L}\{x\} \end{cases} \Rightarrow \begin{cases} [sX(s) - \overbrace{x(0)}^{0}] = -X(s) + Y(s) \\ [sY(s) - \underbrace{y(0)}_{1}] = 2X(s) \end{cases}$$

$$\Rightarrow \begin{cases} X(s)[s+1] - Y(s) = 0 \\ -2X(s) + sY(s) = 1 \end{cases} \Rightarrow \begin{bmatrix} s+1 & -1 \\ -2 & s \end{bmatrix}\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} s & 1 \\ 2 & s+1 \end{bmatrix} \frac{1}{s^2+s-2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{(s+2)(s-1)} \begin{bmatrix} 1 \\ s+1 \end{bmatrix}$$

$$\Rightarrow X(s) = \frac{1}{(s+2)(s-1)} \ \& \ Y(s) = \frac{s+1}{(s+2)(s-1)}.$$

$1 = A(s-1) + B(s+2)$ $\qquad$ $s+1 = A(s-1) + B(s+2)$

$\Rightarrow 1 = 3B \Rightarrow B = 1/3$ $\qquad$ $\Rightarrow 2 = 3B \Rightarrow B = 2/3$

$\& \Rightarrow 1 = -3A \Rightarrow A = -1/3.$ $\qquad$ $\& \Rightarrow -1 = -3A \Rightarrow A = 1/3.$

$$X(s) = \frac{-1}{3(s+2)} + \frac{1}{3(s-1)} \qquad Y(s) = \frac{1}{3(s+2)} + \frac{2}{3(s-1)}.$$

$$\Rightarrow x(t) = -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = -\frac{1}{3}e^{-2t} + \frac{1}{3}e^{t}.$$

$$y(t) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = \frac{1}{3}e^{-2t} + \frac{2}{3}e^{t}.$$

3. $\begin{cases} \frac{dx}{dt} = x - 2y \\ \frac{dy}{dt} = 5x - y \end{cases}$, $x(0) = -1$, $y(0) = 2$.

$$\begin{cases} [sX(s) - \underbrace{x(0)}_{-1}] = X(s) - 2Y(s) \\ [sY(s) - \underbrace{y(0)}_{2}] = 5X(s) - Y(s) \end{cases}$$

$$\Rightarrow \begin{cases} X(s)[s-1] + 2Y(s) = -1 \\ -5X(s) + [s+1]Y(s) = 2 \end{cases} \Rightarrow \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} s-1 & 2 \\ -5 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} s+1 & -2 \\ 5 & s-1 \end{bmatrix} \frac{1}{s^2-1+10} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \frac{1}{s^2+9} \begin{bmatrix} -s-5 \\ 2s-7 \end{bmatrix}$$

$$\Rightarrow X(s) = \frac{-(s+5)}{s^2+9} \quad \& \quad Y(s) = \frac{2s-7}{s^2+9}$$

$$\Rightarrow x(t) = -\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} \;\&\; y(t) = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - \frac{7}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\Rightarrow x(t) = -\cos(3t) + \tfrac{5}{3}\sin(3t) \;\&\; y(t) = 2\cos(3t) - \tfrac{7}{3}\sin(3t).$$

9. $\begin{cases} x'' + y'' = t^2 \\ x'' - y'' = 4t \end{cases}$, $x(0) = 8$, $x'(0) = 0$, $y(0) = 0$, $y'(0) = 0$.

$$\begin{cases} [s^2X(s) - s\overbrace{x(0)}^{8} - \overbrace{x'(0)}^{0}] + [s^2Y(s) - s\overbrace{y(0)}^{0} - \overbrace{y'(0)}^{0}] = \frac{2}{s^3} \\ [s^2X(s) - s\underbrace{x(0)}_{8} - \underbrace{x'(0)}_{0}] - [s^2Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}] = \frac{4}{s^2} \end{cases}$$

$$\Rightarrow \begin{cases} s^2X(s) + s^2Y(s) = \frac{2}{s^3} + 8s \\ s^2X(s) - s^2Y(s) = \frac{4}{s^2} + 8s \end{cases}$$

$$2s^2X(s) = \frac{2}{s^3} + \frac{4}{s^2} + 16s \quad \& \quad Y(s)2s^2 = \frac{2}{s^3} - \frac{4}{s^2}$$

$$\Rightarrow X(s) = \frac{1}{s^5} + \frac{2}{s^4} + \frac{8}{s} \qquad \Rightarrow Y(s) = \frac{1}{s^5} - \frac{2}{s^4}$$

$\Rightarrow x(t) = \frac{1}{4!} t^4 + \frac{2}{3!} t^3 + 8$ & $y(t) = \frac{1}{4!} t^4 - \frac{2}{3!} t^3$.