7.5: The Dirac Delta Function

 # 1, 3, 9, 11: Solve the IVP.

1. $y' - 3y = \delta(t-2),\ y(0) = 0.$

$$sY(s) - \underbrace{y(0)}_{0} - 3Y(s) = \mathcal{L}\{\delta(t-2)\}$$

$$\Rightarrow Y(s)[s-3] = e^{-2s} \Rightarrow Y(s) = e^{-2s}\frac{1}{s-3}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s-3}\right\} = \mathcal{U}(t-2)\,e^{3(t-2)}.$$

3. $y'' + y = \delta(t-2\pi),\ y(0) = 0,\ y'(0) = 1.$

$$s^2Y(s) - \underbrace{sy(0)}_{0} - \underbrace{y'(0)}_{1} + Y(s) = e^{-2\pi s}$$

$$\Rightarrow Y(s) = \frac{e^{-2\pi s}}{s^2+1} + \frac{1}{s^2+1} \Rightarrow y(t) = \mathcal{U}(t-2\pi)\underbrace{\sin(t-2\pi)}_{\sin t} + \sin t.$$

9. $y'' + 4y' + 5y = \delta(t-2\pi),\ y(0) = 0,\ y'(0) = 0.$

$$s^2Y(s) + 4sY(s) + 5Y(s) = e^{-2\pi s}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{e^{-2\pi s}\frac{1}{s^2+4s+5}\right\} = \mathcal{U}(t-2\pi)\,\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}\Big|_{t\to t-2\pi}$$

$$= \mathcal{U}(t-2\pi)\,e^{-2(t-2\pi)}\underbrace{\sin(t-2\pi)}_{\sin t}.$$

11. $y'' + 4y' + 13y = \delta(t-\pi) + \delta(t-3\pi)$, $y(0)=1$, $y'(0)=0$.

$$s^2 Y(s) - s + 4sY(s) - 4 + 13Y(s) = e^{-\pi s} + e^{-3\pi s}$$

$$\Rightarrow Y(s)\left[s^2+4s+13\right] = e^{-\pi s} + e^{-3\pi s} + s + 4$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s} + e^{-3\pi s} + s+4}{(s+2)^2+9}\right\} = \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{1}{(s+2)^2+9}\right\}$$

$$+ \mathcal{L}^{-1}\left\{e^{-3\pi s}\frac{1}{(s+2)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s+2)^2+9}\right\}$$

$$= \mathcal{U}(t-\pi)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}\Big|_{\substack{t\to t-\pi \\ s\to s+2}} + \mathcal{U}(t-3\pi)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}\Big|_{\substack{t\to t-3\pi \\ s\to s+2}}$$

$$+ \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}\Big|_{s\to s+2} + \mathcal{L}^{-1}\left\{\frac{2}{s^2+9}\right\}\Big|_{s\to s+2}$$

$$= \tfrac{1}{3}\mathcal{U}(t-\pi)\,e^{-2(t-\pi)}\sin(3(t-\pi)) + \tfrac{1}{3}\mathcal{U}(t-3\pi)\,e^{-2(t-3\pi)}\sin(3(t-3\pi))$$

$$+ e^{-2t}\cos(3t) + \tfrac{2}{3}e^{-2t}\sin(3t).$$