7.4: Operational Properties II

1, 3, 7, 11, 13, 19, 21, 25, 27, 31, 37, 45

#1, 3, 7: Evaluate.

1. $\mathcal{L}\{t e^{-10t}\} = (-1)\frac{d}{ds}\mathcal{L}\{e^{-10t}\} = -\frac{d}{ds}\frac{1}{s+10} = \frac{1}{(s+10)^2}$.

3. $\mathcal{L}\{t\cos(2t)\} = (-1)\frac{d}{ds}\mathcal{L}\{\cos(2t)\} = -\frac{d}{ds}\frac{s}{s^2+4}$

$= \frac{-(s^2+4)+s(2s)}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}$.

7. $\mathcal{L}\{t e^{2t}\sin(6t)\} = -\frac{d}{ds}\mathcal{L}\{e^{2t}\sin(6t)\} = -\frac{d}{ds}\mathcal{L}\{\sin(6t)\}\Big|_{s\to s-2}$

$= -\frac{d}{ds}\frac{6}{(s-2)^2+36} = -\frac{d}{ds}\frac{6}{s^2-4s+40} = \frac{6[2s-4]}{(s^2-4s+40)^2}$.

#11, 13: Solve the IVP.

11. $y'' + 9y = \cos(3t)$, $y(0) = 2$, $y'(0) = 5$.

$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = \mathcal{L}\{\cos(3t)\}$

$\Rightarrow s^2 Y(s) - s\underbrace{y(0)}_{2} - \underbrace{y'(0)}_{5} + 9Y(s) = \frac{s}{s^2+9}$

$\Rightarrow Y(s)[s^2+9] = \frac{s}{s^2+9} + 2s + 5$

$u = s^2+9$
$\frac{1}{2}du = s\,ds$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2} + \frac{2s+5}{s^2+9}\right\}$ $\quad \int \frac{s}{(s^2+9)^2} = \frac{1}{2}\int u^{-2}\,du = -\frac{1}{2}u^{-1} = -\frac{1}{2}(s^2+9)^{-1}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{d}{ds}\left(\frac{-1}{2(s^2+9)}\right)\right\} + 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{5}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$

$\Rightarrow y(t) = t\,\mathcal{L}^{-1}\left\{\frac{1}{2(s^2+9)}\right\} + 2\cos(3t) + \frac{5}{3}\sin(3t)$

$\Rightarrow y(t) = \frac{t}{6}\sin(3t) + 2\cos(3t) + \frac{5}{3}\sin(3t)$.

13. $y'' + 16y = F(t)$, $y(0) = 0$, $y'(0) = 1$, where $F(t) = \begin{cases}\cos(4t), & 0 \le t < \pi \\ 0, & t \ge \pi.\end{cases}$

$\mathcal{L}\{y''\} + 16\mathcal{L}\{y\} = \mathcal{L}\{\cos(4t) - \cos(4t)\,\mathcal{U}(t-\pi)\}$

$\Rightarrow s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{1} + 16Y(s) = \frac{s}{s^2+16} - e^{-\pi s}\mathcal{L}\{\cos 4(t+\pi)\}$

$\Rightarrow Y(s)[s^2+16] = \frac{s}{s^2+16} - e^{-\pi s}\mathcal{L}\{\cos(4t)\} + 1$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+16)^2}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s}\frac{s}{(s^2+16)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\}$

$u = s^2+16$

$\frac{1}{2}du = s\,ds$

$\int \frac{s}{(s^2+16)^2}\,ds = -\frac{1}{2}\frac{1}{s^2+16}$.

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}\frac{1}{2(s^2+16)}\right\} - \mathcal{U}(t-\pi)\,\mathcal{L}^{-1}\left\{-\frac{d}{ds}\frac{1}{2(s^2+16)}\right\}\Big|_{t\to t-\pi}$$

$$+ \frac{1}{4}\sin(4t)$$

$$\Rightarrow y(t) = \frac{t}{8}\sin(4t) - \frac{t}{8}\sin(4t)\,\mathcal{U}(t-\pi)\Big|_{t\to t-\pi} + \frac{1}{4}\sin(4t)$$

$$\Rightarrow y(t) = \frac{t}{8}\sin(4t) - \frac{(t-\pi)}{8}\sin(4(t-\pi))\,\mathcal{U}(t-\pi) + \frac{1}{4}\sin(4t)$$

<u># 19, 21, 25, 27</u>: Evaluate.

19. $\mathcal{L}\{1 * t^3\} = \mathcal{L}\{1\}\mathcal{L}\{t^3\} = \frac{1}{s}\cdot\frac{3!}{s^4} = \frac{6}{s^5}$.

21. $\mathcal{L}\{e^{-t} * e^{t}\cos t\} = \mathcal{L}\{e^{-t}\}\,\mathcal{L}\{e^{t}\cos t\} = \frac{1}{s+1}\,\mathcal{L}\{\cos t\}\big|_{s\to s-1}$

$= \frac{1}{s+1}\,\frac{(s-1)}{(s-1)^2+1} = \frac{s-1}{(s+1)[(s-1)^2+1]}$.

25. $\mathcal{L}\left\{\int_0^t e^{-\tau}\cos\tau\,d\tau\right\} = \mathcal{L}\{e^{-t}\cos t * 1\} = \mathcal{L}\{\cos t\}\big|_{s\to s+1}\,\frac{1}{s}$

$= \frac{s+1}{[(s+1)^2+1]s}$.

27. $\mathcal{L}\left\{\int_0^t \tau e^{t-\tau}\,d\tau\right\} = \mathcal{L}\{t * e^{t}\} = \mathcal{L}\{t\}\mathcal{L}\{e^{t}\}$

$= \frac{1}{s^2}\frac{1}{s-1} = \frac{1}{s^3-s^2}$.

31. Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s(s-1)}\right\}$.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\,\frac{1}{s-1}\right\} = \mathcal{L}^{-1}\{\mathcal{L}\{1\}\,\mathcal{L}\{e^{t}\}\} = \mathcal{L}^{-1}\mathcal{L}\{1 * e^{t}\}$$

$$= 1 * e^{t} = \int_0^t e^{\tau}\,d\tau = e^{\tau}\Big|_0^t = e^{t} - 1.$$

#37, 45: Solve.

37. $F(t) + \int_0^t (t-\tau) F(\tau)\, d\tau = t$

$\Rightarrow \mathcal{L}\{F(t)\} + \mathcal{L}\{F(t) * t\} = \mathcal{L}\{t\}$

$\Rightarrow F(s) + F(s)\frac{1}{s^2} = \frac{1}{s^2}$

$\Rightarrow s^2 F(s) + F(s) = 1 \Rightarrow F(s) = \frac{1}{s^2+1} \Rightarrow F(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$\Rightarrow F(t) = \sin t.$

45. $y'(t) = 1 - \sin t - \int_0^t y(\tau)\, d\tau, \; y(0) = 0.$

$sY(s) - \underbrace{y(0)}_{0} = \frac{1}{s} - \frac{1}{s^2+1} - \mathcal{L}\{y(t) * 1\}$

$\Rightarrow sY(s) = \frac{1}{s} - \frac{1}{s^2+1} - Y(s)\frac{1}{s}$

$\Rightarrow Y(s)[s^2+1] = 1 - \frac{s}{(s^2+1)}$

$\int \frac{s}{(s^2+1)^2}\, ds = -\frac{1}{2}\frac{1}{s^2+1}.$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

$\Rightarrow y(t) = \sin t - \frac{t}{2}\sin t.$