7.3: Operational Properties I

\# 1, 5, 11, 13, 17, 27, 37, 45, 49, 55, 63, 67

\#1, 5, 11, 13, 17: Find either $F(s)$ or $f(t)$, as indicated.

1. $\mathcal{L}\{te^{10t}\}$.

Recall: $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$.

So, $\mathcal{L}\{te^{10t}\} = \mathcal{L}\{t\}_{s \to s-10} = \dfrac{1}{(s-10)^2}$.

5. $\mathcal{L}\{t(e^t + e^{2t})^2\}$

$= \mathcal{L}\{t(e^{2t} + 2e^{3t} + e^{4t})\} = \mathcal{L}\{te^{2t}\} + 2\mathcal{L}\{te^{3t}\} + \mathcal{L}\{te^{4t}\}$

$= \mathcal{L}\{t\}_{s \to s-2} + 2\mathcal{L}\{t\}_{s \to s-3} + \mathcal{L}\{t\}_{s \to s-4}$

$= \dfrac{1}{(s-2)^2} + \dfrac{2}{(s-3)^2} + \dfrac{1}{(s-4)^2}$.

11. $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^3}\right\}$

Recall: $\mathcal{L}^{-1}\{F(s-a)\} = \mathcal{L}^{-1}\{F(s)|_{s \to s-a}\} = e^{at}f(t)$.

So, $\mathcal{L}^{-1}\left\{\dfrac{1}{(s+2)^3}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^3}\Big|_{s \to s+2}\right\} = \dfrac{1}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{s^3}\Big|_{s \to s+2}\right\}$

$= \frac{1}{2}t^2e^{-2t}$.

13. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\}$.

$= \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\Big|_{s\to s-3}\right\} = \sin t\, e^{3t}$.

17. $\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$

$= e^{-t} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\Big|_{s\to s+1}\right\} = e^{-t} - t e^{-t}$.

27. Use Laplace transforms to solve the IVP $y''-6y'+13y=0$, $y(0)=0$, $y'(0)=-3$.

$\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 13\mathcal{L}\{y\} = \mathcal{L}\{0\}$

$\Rightarrow \left[s^2 Y(s) - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=-3}\right] - 6\left[sY(s) - \underbrace{y(0)}_{=0}\right] + 13Y(s) = 0$

$\Rightarrow Y(s)[s^2-6s+13] = -3$

$\Rightarrow Y(s) = \frac{-3}{s^2-6s+13} = \frac{-3}{(s-3)^2+4}$

$\Rightarrow y(t) = -3\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\Big|_{s\to s-3}\right\} = -\frac{3}{2}\sin(2t)\, e^{3t}$.

#37, 45: Find either $F(s)$ or $f(t)$, as indicated.

37. $\mathcal{L}\{(t-1)\mathcal{U}(t-1)\}$.

Recall: $\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$, where $F(s)=\mathcal{L}\{f(t)\}$.

So, here $\mathcal{L}\{(t-1)\mathcal{U}(t-1)\} = e^{-s}\mathcal{L}\{t\} = \frac{e^{-s}}{s^2}$.

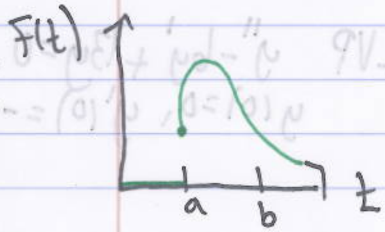
45. $\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$.

<u>Recall</u>: $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a)$.

So, here $a=\pi$ & $F(s) = \frac{1}{s^2+1} \Rightarrow f(t) = \sin t$

$\Rightarrow \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = \sin(t-\pi)\,\mathcal{U}(t-\pi) = -\sin t\,\mathcal{U}(t-\pi)$.

49. Match the graph with one of the given functions.



This corresponds to $\boxed{c}$: $f(t)\,\mathcal{U}(t-a)$.

55. Write $f(t) = \begin{cases} 2, & 0 \le t < 3 \\ -2, & t \ge 3 \end{cases}$ in terms of unit step functions & find the Laplace transform.

$f(t) = 2 - 4\,\mathcal{U}(t-3)$.

$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{1\} - 4\mathcal{L}\{\mathcal{U}(t-3)\} = \frac{2}{s} - 4\frac{e^{-3s}}{s}$.

<u>#63, 67</u>: Use the Laplace transform to solve the given IVP.

63. $y' + y = f(t)$, $y(0)=0$, where $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 5, & t \ge 1 \end{cases}$.

$f(t) = 5\,\mathcal{U}(t-1)$.

$\mathcal{L}\{y'\} + \mathcal{L}\{y\} = 5\mathcal{L}\{\mathcal{U}(t-1)\}$

$[sY(s) - y(0)] + Y(s) = 5\frac{e^{-s}}{s}$

Recall: $\mathcal{L}\{g(t)\,\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$.

$$Y(s)[s+1] = \frac{5e^{-s}}{s}$$

$$y(t) = 5\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+1)}\right\} = 5(1-e^{-(t-1)})\,\mathcal{U}(t-1).$$

$a=1$

$$F(s) = \frac{1}{s(s+1)} = \frac{A_1}{s} + \frac{A_2}{s+1} \Rightarrow 1 = A_1(s+1) + A_2 s$$

$$\Rightarrow 1 = (A_1+A_2)s + A_1$$

$$\Rightarrow A_1 = 1 \;\&\; A_2 = -1.$$

$$F(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = 1 - e^{-t}$$

67. $y'' + 4y = \sin t\,\mathcal{U}(t-2\pi)$, $y(0)=1$, $y'(0)=0$.

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin t\,\mathcal{U}(t-2\pi)\}$$

$$\Rightarrow [s^2Y(s) - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0}] + 4Y(s) = e^{-2\pi s}\mathcal{L}\{\underbrace{\sin(t+2\pi)}_{\sin(t)}\}$$

$$\Rightarrow Y(s)[s^2+4] = \frac{e^{-2\pi s}}{s^2+1} + s$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{(s^2+1)(s^2+4)}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{B_1 s + C_1}{(s^2+1)} + \frac{B_2 s + C_2}{(s^2+4)} \Rightarrow 1 = B_1 s^3 + 4B_1 s + C_1 s^2 + 4C_1 + B_2 s^3 + B_2 s + C_2 s^2 + C_2$$

$$\Rightarrow 1 = [B_1+B_2]s^3 + [C_1+C_2]s^2 + [4B_1+B_2]s + [4C_1+C_2]$$

$B_1 + B_2 = 0$

$-4B_1 - B_2 = 0$

$-3B_1 = 0 \Rightarrow B_1 = B_2 = 0.$

$C_1 = -C_2$. $3C_1 = 1 \Rightarrow C_1 = \frac{1}{3}$ & $C_2 = \frac{-1}{3}$.

$$\Rightarrow y(t) = \cos(2t) + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{e^{-2\pi s}}{s^2+4}\right\}$$

$$= \cos(2t) + \frac{1}{3}\sin(t-2\pi)\,\mathcal{U}(t-2\pi) - \frac{1}{6}\sin(2t-4\pi)\,\mathcal{U}(t-2\pi)$$