7.2: Inverse Transforms & Transforms of Derivatives

#3, 11, 15, 17, 31, 37

#3, 11, 15, 17: Using the Laplace transform table, Find the given inverse Laplace transform.

3. $\mathcal{L}^{-1}\left\{\frac{1}{s^2}-\frac{48}{s^5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 48\,\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$

$= t - \frac{48}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = t - \frac{48}{24}t^4 = t - 2t^4.$

11. $\mathcal{L}^{-1}\left\{\frac{5}{s^2+49}\right\} = \frac{5}{7}\mathcal{L}^{-1}\left\{\frac{7}{s^2+7^2}\right\} = \frac{5}{7}\sin(7t).$

15. $\mathcal{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} - \frac{6}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$

$= 2\cos(3t) - 2\sin(3t).$

17. $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+3)}\right\}.$

$\frac{1}{s(s+3)} = \frac{A_1}{s} + \frac{A_2}{s+3} \Rightarrow 1 = A_1(s+3) + A_2 s \Rightarrow 1 = (A_1+A_2)s + 3A_1$

$\Rightarrow 3A_1 = 1$ & $A_1 + A_2 = 0 \Rightarrow A_1 = \frac{1}{3}$ & $A_2 = -\frac{1}{3}.$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$

$= \frac{1}{3} - \frac{1}{3}e^{-3t}.$

#31, 37: Use the Laplace Transform to solve the IVP.

31. $y' - y = 1, \quad y(0) = 0.$

$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = \mathcal{L}\{1\}$

$sY(s) - \underbrace{y(0)}_{=0} - Y(s) = 1/s$

$Y(s)[s-1] = 1/s$

$Y(s) = \dfrac{1}{s(s-1)}$

$y(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s(s-1)}\right\}$

$y(t) = -\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\}$

$y(t) = -1 + e^{t}.$

$\dfrac{1}{s(s-1)} = \dfrac{A_1}{s} + \dfrac{A_2}{s-1}$

$\Rightarrow 1 = A_1(s-1) + A_2 s$

$\Rightarrow 1 = (A_1 + A_2)s - A_1$

$\Rightarrow A_1 = -1$ & $-1 + A_2 = 0 \Rightarrow A_2 = 1.$

37. $y'' + y = \sqrt{2}\sin(\sqrt{2}t), \quad y(0) = 10, \quad y'(0) = 0.$

$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \sqrt{2}\,\mathcal{L}\{\sin(\sqrt{2}t)\}$

$\Rightarrow [s^2 Y(s) - s\underbrace{y(0)}_{10} - \underbrace{y'(0)}_{0}] + Y(s) = \sqrt{2}\,\dfrac{\sqrt{2}}{s^2+2}$

$\Rightarrow Y(s)[s^2+1] - 10s = \dfrac{2}{s^2+2}$

$\Rightarrow Y(s) = \dfrac{2}{(s^2+1)(s^2+2)} + \dfrac{10s}{s^2+1}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\dfrac{2}{(s^2+1)(s^2+2)}\right\} + 10\,\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\dfrac{2}{(s^2+1)(s^2+2)}\right\} + 10\cos t.$

$$\frac{2}{(s^2+1)(s^2+2)} = \frac{B_1 s + C_1}{(s^2+1)} + \frac{B_2 s + C_2}{(s^2+2)}$$

$$\Rightarrow 2 = (B_1 s + C_1)(s^2+2) + (B_2 s + C_2)(s^2+1)$$

$$\Rightarrow 2 = B_1 s^3 + 2B_1 s + C_1 s^2 + 2C_1 + B_2 s^3 + B_2 s + C_2 s^2 + C_2$$

$$\Rightarrow 2 = s^3 \underbrace{(B_1 + B_2)}_{=0} + s^2 \underbrace{(C_1 + C_2)}_{=0} + s \underbrace{(2B_1 + B_2)}_{=0} + \underbrace{(2C_1 + C_2)}_{=1}$$

$$\left.\begin{aligned} B_1 + B_2 &= 0 \\ 2B_1 + B_2 &= 0 \end{aligned}\right\} \Rightarrow B_1 = 0 \ \& \ B_2 = 0. \quad C_1 = -C_2. \quad -2C_2 + C_2 = 2$$

$$\Rightarrow -C_2 = 2$$

$$\Rightarrow C_2 = -2$$

$$\& \ C_1 = 2.$$

$$\therefore \frac{2}{(s^2+1)(s^2+2)} = \frac{2}{s^2+1} - \frac{2}{s^2+2}.$$

$$\therefore y(t) = 10\cos t + 2\sin t - \frac{2}{\sqrt{2}}\sin(\sqrt{2}\,t).$$