6.3: Solutions About Singular Points

#1, 3, 5, 13, 15, 23, 25, 27, 29

#1, 3, 5: Classify each singular point as regular or irregular.

1. $x^3 y'' + 4x^2 y' + 3y = 0.$

Here $P(x) = \frac{4}{x}$ & $Q(x) = \frac{3}{x^3}$ $\Rightarrow$ this eq'n has one singular pt $x = 0$, & it's irregular since it appears to the 3rd power in $Q(x)$.

3. $(x^2-9)^2 y'' + (x+3)y' + 2y = 0.$

$P(x) = \frac{x+3}{(x+3)^2(x-3)^2} = \frac{1}{(x+3)(x-3)^2}$ $\Rightarrow$ $x=3$ irregular singular point.

$Q(x) = \frac{2}{(x+3)^2(x-3)^2}$ $\Rightarrow$ $x = -3$ regular singular pt.

5. $(x^3+4x)y'' - 2xy' + 6y = 0.$

$P(x) = \frac{-2x}{x(x^2+4)} = \frac{-2}{x^2+4} = \frac{-2}{(x+2i)(x-2i)}.$

$Q(x) = \frac{6}{x(x^2+4)}$ $\Rightarrow$ $x = 0, 2i, \text{ \& } -2i$ are regular singular points.

13. $x=0$ is a regular singular pt of $x^2y''+(\frac{5}{3}x+x^2)y' - \frac{1}{3}y=0$. Use the indicial eq'n to find the indicial roots of the singularity & w/out solving discuss the number of series solutions you would expect to find using the method of Frobenius.

$$P(x)=\frac{\frac{5}{3}x+x^2}{x^2}=\frac{5}{3x}+1.$$

$$a_0=\lim_{x\to 0} xP(x)=\lim_{x\to 0}\frac{5}{3}+x=\frac{5}{3}.$$

$$Q(x)=\frac{-1}{3x^2}.\quad b_0=\lim_{x\to 0}x^2Q(x)=\lim_{x\to 0}\frac{-1}{3}=-\frac{1}{3}.$$

$$r(r-1)+a_0r+b_0=0$$

$$\Leftrightarrow r(r-1)+\frac{5}{3}r-\frac{1}{3}=0 \Leftrightarrow r^2+\frac{2}{3}r-\frac{1}{3}=0$$

$$\Leftrightarrow 3r^2+2r-1=0 \Leftrightarrow (r+1)(r-\tfrac{1}{3})=0$$

$(r+3)(r-1)$ $\Leftrightarrow r_2=-1$ or $r_1=\frac{1}{3}$. & $r_1\neq r_2$, $r_1-r_2=\frac{4}{3}$ not an

$(r+1)(3r-1)$

$r=-1$ $r=\frac{1}{3}$ integer $\Rightarrow \exists$ 2 lin. ind. solutions to this DE of the form $y_1=\sum_{n=0}^{\infty}c_nx^{n+r_1}$ & $y_2=\sum_{n=0}^{\infty}b_nx^{n+r_2}$

$\Rightarrow$ we expect to find 2 series solutions using the method of Frobenius.

#15.23: $x=0$ is a regular pt. Show the indicial roots of the singularity don't differ by an integer & use the method of Frobenius to obtain 2 linearly independent series solutions about $x=0$. Form the general solution on $(0,\infty)$.

15. $2xy'' - y' + 2y = 0.$

$r(r-1) + a_0 r + b_0 = 0$

$P(x) = \frac{-1}{2x} \Rightarrow \lim_{x\to 0} xP(x) = -\frac{1}{2}.$ $\Leftrightarrow r^2 - r - \frac{1}{2}r = 0 \Leftrightarrow r_2 = 0$ or $r_1 = \frac{3}{2}$

$Q(x) = \frac{2}{2x} \Rightarrow \lim_{x\to 0} x^2 Q(x) = 0.$ $r_1 - r_2 = \frac{3}{2}$ not integer $\Rightarrow$ will get 2 lin. ind. solutions.

$y = \sum_{n=0}^{\infty} c_n x^{n+r},\quad y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1},$

$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}.$

$$0 = 2xy'' - y' + 2y = x^r \Big[\overbrace{2\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n-1}}^{k=n} - \overbrace{\sum_{n=0}^{\infty} c_n (n+r) x^{n-1}}^{k=n} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{k-1=n} \Big]$$

$$= x^r \Big[2\sum_{k=0}^{\infty} c_k (k+r)(k+r-1) x^{k-1} - \sum_{k=0}^{\infty} c_k (k+r) x^{k-1} + 2\sum_{k=1}^{\infty} c_{k-1} x^{k-1} \Big]$$

$$= x^r \Big[2c_0 r(r-1)x^{-1} - c_0 r x^{-1} + \sum_{k=1}^{\infty} \big[2c_k (k+r)(k+r-1) - c_k(k+r) + 2c_{k-1} \big] x^{k-1} \Big]$$

$$\Rightarrow c_k = \frac{-2c_{k-1}}{2(k+r)(k+r-1) - (k+r)} = \frac{-2c_{k-1}}{[2(k+r-1)-1](k+r)}.$$

$r_2 = 0$: $c_k = \frac{-2c_{k-1}}{(2k-3)k} \Rightarrow c_1 = \frac{-2}{-1} c_0 = 2c_0,\; c_2 = \frac{-2}{2} c_1 = -2c_0,$

$c_3 = \frac{-2}{9} c_2 = \frac{4}{9} c_0$, etc.

So, $y_1 = c_0 + 2c_0 x - 2c_0 x^2 + \frac{4}{9} c_0 x^3 + \cdots$.

$\underline{r_1 = \frac{3}{2}}$: $c_K = \dfrac{-2}{2K(K+\frac{3}{2})} c_{K-1} = \dfrac{-2\, c_{K-1}}{K(2K+3)}$

$\Rightarrow c_1 = \dfrac{-2c_0}{5} = \dfrac{-2}{5}c_0$, $c_2 = \dfrac{-2}{2(7)} c_1 = \dfrac{-1}{7}\,\dfrac{-2}{5} c_0 = \dfrac{2}{35} c_0$,

$c_3 = \dfrac{-2}{3(9)} c_2 = \dfrac{-2}{27}\left(\dfrac{2}{35}\right) c_0 = \dfrac{-4}{945} c_0$.

35
27
245
700
945

So, $y_2 = x^{3/2} c_0 - \frac{2}{5} c_0 x^{\frac{3}{2}+1} + \frac{2c_0}{35} x^{2+\frac{3}{2}} - \frac{4}{945} c_0 x^{3+\frac{3}{2}} + \cdots$.

∴ A general solution is:

$$y = c_1\left(1 + 2x - 2x^2 + \tfrac{4}{9}x^3 + \cdots\right) + c_2 x^{3/2}\left(1 - \tfrac{2}{5}x + \tfrac{2}{35}x^2 - \tfrac{4}{945}x^3 + \cdots\right).$$

23. $9x^2 y'' + 9x^2 y' + 2y = 0$.

$$0 = 9\underbrace{\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r}}_{K=n} + 9\underbrace{\sum_{n=0}^{\infty} (n+r)c_n x^{n+r+1}}_{K=n+1} + 2\underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{n=K}$$

$$= 9\sum_{K=0}^{\infty} (K+r)(K+r-1)c_K x^{K+r} + 9\sum_{K=1}^{\infty} (K-1+r)c_{K-1} x^{K+r} + 2\sum_{K=0}^{\infty} c_K x^{K+r}$$

$$= x^r\left[9r(r-1)+2\right]c_0 + \sum_{k=1}^{\infty} x^k\left[9(k+r)(k+r-1)+2\right]c_k + 9(k-1+r)c_{k-1}$$

$$\Rightarrow 9r^2 - 9r + 2 = (3r-2)(3r-1) = 0 \Rightarrow r_1 = 2/3 \text{ or } r_2 = 1/3.$$

$(9r-6)(9r-3)$

$(3r-2)(3r-1)$

$r_1 - r_2 = \frac{1}{3}$ not integer.

$$c_k = \frac{-9(k-1+r)\, c_{k-1}}{9(k+r)(k+r-1)+2}.$$

$\underline{r_1 = 2/3}$: $c_k = -\dfrac{(9k-3)}{(3k+2)(3k-1)+2}$ $\Rightarrow c_1 = \dfrac{-6}{12}c_0 = \dfrac{-1}{2}c_0$,

$c_2 = \dfrac{-15}{42}c_1 = \dfrac{5}{28}c_0$,

$c_3 = \dfrac{-24}{90}c_2 = \dfrac{-1}{18}\cdot\dfrac{6}{7}c_0 = \dfrac{-1}{21}c_0$, etc.

$$\therefore y_1 = x^{2/3}c_0 - \tfrac{1}{2}c_0 x^{1+2/3} + \frac{5}{28}c_0 x^{2+2/3} - \frac{1}{21}c_0 x^{3+2/3} + \dots$$

$\underline{r_2 = 1/3}$: $c_k = \dfrac{-9(k-2/3)\, c_{k-1}}{9(k+1/3)(k-2/3)+2}$

$$= \frac{-9k+6\; c_{k-1}}{(3k+1)(3k-2)+2}$$

$\Rightarrow c_1 = \dfrac{-3}{6}c_0 = \dfrac{-1}{2}c_0$, $c_2 = \dfrac{-12}{30}c_1 = \dfrac{6}{30}c_0 = \dfrac{1}{5}c_0$,

$c_3 = \dfrac{-21}{72}c_2 = \dfrac{-21}{72\cdot 5}c_0 = \dfrac{-7}{120}c_0$, etc.

24
5
$\overline{120}$

$$\therefore y_2 = x^{1/3}c_0 - \tfrac{1}{2}c_0 x^{1+1/3} + \tfrac{1}{5}c_0 x^{2+1/3} - \frac{7}{120}c_0 x^{3+1/3} + \dots$$

$\therefore$ A general solution is:

$$y = c_1 x^{2/3}\left[1 - \tfrac{1}{2}x + \tfrac{5}{28}x^2 - \tfrac{1}{21}x^3 + \dots\right]$$
$$+ c_2 x^{1/3}\left[1 - \tfrac{1}{2}x + \tfrac{1}{5}x^2 - \tfrac{7}{120}x^3 + \dots\right].$$

#25, 27, 29: $x=0$ is a regular singular pt of the DE's. Show the indicial roots differ by an integer & use method of Frobenius to obtain as many solutions as possible.

25. $xy'' + 2y' - xy = 0.$

$$\underbrace{\sum_{n=0}^{\infty} c_n(n+r)(n+r-1)\, x^{n+r-1}}_{k=n} + \underbrace{2\sum_{n=0}^{\infty}(n+r)c_n\, x^{n+r-1}}_{k=n} - \underbrace{\sum_{n=0}^{\infty} c_n\, x^{n+r+1}}$$

$n+1 = k-1$
$\Rightarrow k = n+2$

$$= \sum_{k=0}^{\infty}\left[c_k(k+r)(k+r-1) + 2c_k(k+r)\right]x^{k+r-1} - \sum_{k=2}^{\infty} c_{k-2}\, x^{k-1+r}$$

$$= x^r c_0\left[r(r-1) + 2r\right] + x^r c_1\left[(r+1)r + 2(r+1)\right]$$

$$+ x^r \sum_{k=2}^{\infty}\left[\left[(k+r)(k+r-1) + 2(k+r)\right]c_k - c_{k-2}\right]x^{k-1}$$

$$\Rightarrow \underbrace{r^2 + r = 0}_{\text{indicial eq}^n} \;\&\; (r^2+3r+2)c_1 = 0 \;\&\; c_k = \frac{c_{k-2}}{(k+r)(k+r+1)}$$

$\Rightarrow r_1 = 0,\ r_2 = -1.$
$r_1 - r_2 = 1$ integer.
So may get one or 2 distinct solutions using this method.

$r=0$: $c_1 = 0$,

$$c_k = \frac{c_{k-2}}{k(k+1)},\ k \geq 2$$

$$\Rightarrow c_2 = \underbrace{\frac{c_0}{6}}_{3!},\ \underbrace{c_3 = 0}_{\text{odd \#'s will be zero}},\ c_4 = \frac{c_2}{20} = \frac{c_0}{5\cdot 4\cdot 3\cdot 2}$$

$c_6 = \frac{c_4}{6\cdot 7} = \frac{c_0}{7!}, \ldots, c_{2n} = \frac{c_0}{(2n+1)!}$.

So, $y_1 = c_0 x^0 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n} = c_0 \frac{\sinh x}{x}$.

$\underline{r=-1}$: $(1-3+2)c_1 = 0 \Rightarrow c_1 = 0$. $c_K = \frac{c_{K-2}}{(K-1)K}$

$\Rightarrow c_2 = \frac{c_0}{2!}, \ c_3 = c_5 = c_7 = \cdots = 0,$

$c_4 = \frac{c_2}{4\cdot 3} = \frac{c_0}{4!}, \ldots, c_{2n} = \frac{c_0}{(2n)!}$.

So, $y_2 = c_0 x^{-1} \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} = c_0 \frac{\cosh x}{x}$.

$\therefore$ A general solution is: $y = C_1 \frac{\sinh x}{x} + C_2 \frac{\cosh x}{x}$.

27. $xy'' - xy' + y = 0$.

$$0 = \underbrace{\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-1}}_{K=n} - \underbrace{\sum_{n=0}^{\infty} (n+r) c_n x^{n+r}}_{K-1=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{K-1=n}$$

$$= \sum_{K=0}^{\infty} (K+r)(K+r-1) c_K x^{K+r-1} + \sum_{K=1}^{\infty} [-(K-1+r)+1] c_{K-1} x^{K-1+r}$$

$$= x^{r-1} c_0 (r(r-1)) + x^r \sum_{K=1}^{\infty} [(K+r)(K+r-1)c_K - (K+r-2)c_{K-1}] x^{K-1}$$

$\Rightarrow r_1 = 1, r_2 = 0$. $r_1 - r_2 = 1$ integer $\Rightarrow$ will get one of 2 solutions here.

$\underline{r_1 = 1}$: $c_K = \dfrac{(K-1)c_{K-1}}{K(K+1)} \Rightarrow c_1 = 0 \Rightarrow c_2 = c_3 = \dots = 0$.

$\therefore\ y_1 = c_0 \cdot x$ is a solution.

$\underline{r_2 = 0}$: Plugging this into our recurrence relation gives:

$c_K = \dfrac{K-2}{K(K-1)} c_{K-1}$, but plugging $K=1$ into this doesn't work...

so we only get one solution using the Frobenius method.

[other solution looks like $y_2 = C\underbrace{y_1}_{x}\ln x + \sum_{n=0}^{\infty} b_n x^n$, b_0].

29. $xy'' + (1-x)y' - y = 0$.

$$0 = \underbrace{\sum_{n=0}^{\infty} c_n (n+r)(n+r-1)\, x^{n+r-1}}_{K=n} + \underbrace{\sum_{n=0}^{\infty} c_n (n+r)\, x^{n+r-1}}_{K=n} - \underbrace{\sum_{n=0}^{\infty} c_n (n+r)\, x^{n+r}}_{K-1=n}$$

$$- \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r}}_{K-1=n}$$

$$= \sum_{K=0}^{\infty} c_K \left[(K+r)(K+r-1) + (K+r)\right] x^{K+r-1} - \sum_{K=1}^{\infty} c_{K-1}\left[(K-1+r)+1\right] x^{K-1+r}$$

$$= x^{r-1} c_0 \underbrace{\left(r(r-1)+r\right)}_{\text{indicial eq}^n} + x^r \sum_{K=1}^{\infty} x^{K-1}\left[c_K\left((K+r)(K+r)\right) - c_{K-1}(K+r)\right]$$

$\Rightarrow r_1 = r_2 = 0 \Rightarrow$ we'll only get one solution using this method [the other one looks like $y_1 \ln x + \sum_{n=1}^{\infty} b_n x^n$].

$\underline{r_1 = 0}$: $c_K = \dfrac{K}{K^2} c_{K-1} = \dfrac{1}{K} c_{K-1}$

$\Rightarrow c_1 = c_0,\quad c_2 = \frac{1}{2} c_0,\quad c_3 = \frac{1}{3.2} c_0,\quad c_4 = \frac{1}{4!} c_0, \ldots, c_n = \frac{c_0}{n!}.$

$\therefore\ y_1 = c_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n = c_0 e^x.$