6.2: Solutions About Ordinary Points

#1, 7, 13, 19, 21, 23, 25, 26

1. Find the min. rad. of convergence of a power series solution of $(x^2-25)y'' + 2xy' + y = 0$ about the ordinary pt (a) $x=0$, (b) $x=1$.

Here $P(x) = \dfrac{2x}{x^2-25} = \dfrac{2x}{(x-5)(x+5)} \Rightarrow x = \pm 5$ singular pts.

$Q(x) = \dfrac{1}{x^2-25}$.

$\boxed{a}$ $\sqrt{(5-0)^2} = 5$. $\therefore R = 5$.

$\boxed{b}$ $\sqrt{(5-1)^2} = \sqrt{4^2} = 4$. $\therefore R = 4$.

#7, 13: Find 2 power series solutions about the ordinary pt $x=0$.

7. $y'' - xy = 0$.

$y = \sum_{n=0}^{\infty} c_n x^n$, $y' = \sum_{n=1}^{\infty} c_n n x^{n-1}$, $y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}$.

$$0 = y'' - xy = \underbrace{\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1}$$

$$= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k$$

$$= 2c_2 + \sum_{k=1}^{\infty}\left[c_{k+2}(k+2)(k+1) - c_{k-1}\right]x^k$$

$\Rightarrow 2c_2 = 0$ & $c_{k+2}(k+2)(k+1) - c_{k-1} = 0$

$\Rightarrow c_2 = 0$ & $c_{k+2} = \dfrac{c_{k-1}}{(k+2)(k+1)}$, $k \geq 1$

$\Rightarrow c_3 = \dfrac{c_0}{6}$ & $c_4 = \dfrac{c_1}{12}$ & $c_5 = \dfrac{c_2}{20} = 0$

& $c_6 = \dfrac{c_3}{30} = \dfrac{c_0}{6 \cdot 30}$ & $c_7 = \dfrac{c_4}{42} = \dfrac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}$

& $c_8 = 0$ & $c_9 = \dfrac{c_6}{72} = \dfrac{c_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}$, etc.

So, $y = c_0 + c_1 x + \frac{c_0}{6} x^3 + \frac{c_1}{12} x^4 + \frac{c_0}{6 \cdot 30} x^6 + \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3} x^7 + \frac{c_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} x^9 + \cdots$.

Random choices for c_0 & c_1

Choosing $c_0 = 1$ & $c_1 = 0$ we obtain a solution:

$$y_1 = 1 + \frac{1}{6} x^3 + \frac{1}{6 \cdot 30} x^6 + \frac{1}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2} x^9 + \cdots.$$

Choosing $c_0 = 0$, $c_1 = 1$ we obtain a solution:

$$y_2 = x + \frac{1}{12} x^4 + \frac{1}{7 \cdot 6 \cdot 4 \cdot 3} x^7 + \cdots.$$

13. $(x-1) y'' + y' = 0.$

$$0 = (x-1) y'' + y' = \underbrace{\sum_{n=2}^{\infty} c_n n(n-1) x^{n-1}}_{k=n-1} - \underbrace{\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}}_{k=n-2}$$

$$+ \underbrace{\sum_{n=1}^{\infty} c_n n x^{n-1}}_{k=n-1}$$

$$= \sum_{k=1}^{\infty} c_{k+1}(k+1)(k)x^k - \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k$$

$$+ \sum_{k=0}^{\infty} c_{k+1}(k+1)x^k$$

$$= -2c_2 + c_1 + \sum_{k=1}^{\infty}\left[c_{k+1}k(k+1) - c_{k+2}(k+2)(k+1) + c_{k+1}(k+1)\right]x^k$$

$$\Rightarrow c_1 - 2c_2 = 0 \quad \& \quad c_{k+2} = \frac{[k(k+1) + (k+1)]}{(k+2)(k+1)} c_{k+1}$$

$$\Rightarrow c_2 = \tfrac{1}{2}c_1 \quad \& \quad c_{k+2} = \frac{k+1}{k+2} c_{k+1}$$

$$\Rightarrow c_3 = \frac{2}{3}c_2 = \frac{1}{3}c_1 \quad \& \quad c_4 = \frac{3}{4}c_3 = \frac{1}{4}c_1$$

$$\& \quad c_5 = \frac{4}{5}c_4 = \frac{1}{5}c_1, \text{ etc.} \Rightarrow c_k = \frac{1}{k}c_1 \text{ for } k \geq 1.$$

So, $y = c_0 + c_1 x + c_2 x^2 + \cdots = c_0 + \sum_{n=1}^{\infty} \frac{1}{n} c_1 x^n$.

So, 2 solutions would be:

$$y_1 = c_0 \quad \& \quad y_2 = c_1\left[\sum_{n=1}^{\infty} \frac{1}{n} x^n\right]$$

could choose any constants here.

#19, 21: Use the power series method to solve the given IVP.

19. $(x-1)y'' - xy' + y = 0, \quad y(0) = -2, \quad y'(0) = 6.$

$y = \sum_{n=0}^{\infty} c_n x^n$. $y(0) = -2 \Rightarrow c_0 = -2$.

$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$. $y'(0) = 6 \Rightarrow c_1 = 6$.

$$0 = (x-1)y'' - xy' + y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1}}_{K=n-1} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{K=n-2}$$
$$- \underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{K=n} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{K=n}$$

$$= \sum_{K=1}^{\infty} K(K+1) c_{K+1} x^K - \sum_{K=0}^{\infty} (K+2)(K+1) c_{K+2} x^K$$

$$- \sum_{K=1}^{\infty} K c_K x^K + \sum_{K=0}^{\infty} c_K x^K$$

$$= -2c_2 + c_0 + \sum_{K=1}^{\infty} x^K \left[K(K+1)c_{K+1} - (K+2)(K+1)c_{K+2} - K c_K + c_K\right]$$

$$\Rightarrow c_2 = \tfrac{1}{2} c_0 \quad \& \quad c_{K+2} = \frac{K(K+1)c_{K+1} + (1-K)c_K}{(K+2)(K+1)}$$

$\Rightarrow c_2 = \frac{1}{2}(-2) = -1$.

$$c_3 = \frac{2c_2}{6} = -\frac{1}{3}. \quad c_4 = \frac{6c_3 - c_2}{12} = \frac{-2+1}{12} = \frac{-1}{\underbrace{12}_{4\cdot 3}}.$$

$$c_5 = \frac{3\cdot 4 c_4 - 2c_3}{5\cdot 4} = \frac{-1 + \frac{2}{3}}{20} = \frac{-1}{\underbrace{60}_{5\cdot 4\cdot 3}}.$$

$$c_6 = \frac{20c_5 - 3c_4}{6\cdot 5} = \frac{-\frac{1}{3} + \frac{1}{4}}{30} = \frac{-1}{\underbrace{12\cdot 30}_{6\cdot 5\cdot 4\cdot 3}}.$$

So, $c_0 = -2$, $c_1 = 6$, & $c_K = \frac{-2}{K!}$.

So, $y = -2 + 6x - 2\sum_{K=2}^{\infty} \frac{1}{K!} x^K = -2\sum_{K=0}^{\infty} \frac{1}{K!} x^K + 8x$

$= -2e^x + 8x$.

21. $y'' - 2xy' + 8y = 0$, $y(0) = 3$, $y'(0) = 0$.

$y(0) = 3 \Rightarrow c_0 = 3$.
$y'(0) = 0 \Rightarrow c_1 = 0$.

$$0 = y'' - 2xy' + 8y = \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{K=n-2} - 2\underbrace{\sum_{n=1}^{\infty} n c_n x^n}_{K=n} + 8\underbrace{\sum_{n=0}^{\infty} c_n x^n}_{K=n}$$

$$= \sum_{K=0}^{\infty} c_{K+2}(K+2)(K+1)x^K - 2\sum_{K=1}^{\infty} K c_K x^K + 8\sum_{K=0}^{\infty} c_K x^K$$

$$= 2c_2 + 8c_0 + \sum_{K=1}^{\infty} \left[(K+2)(K+1)c_{K+2} - 2Kc_K + 8c_K\right] x^K$$

$\Rightarrow c_2 = -4c_0$ & $c_{K+2} = \frac{-(-2K+8)}{(K+2)(K+1)} c_K$

$\Rightarrow c_2 = -12$

& $\underbrace{c_3 = \frac{-(-2+8)}{3 \cdot 2} c_1 = 0}_{\text{all odd terms will be zero}}$, & $c_4 = \frac{-(4)}{12} c_2 = 4$

& $\underbrace{c_6 = \frac{-(0)}{6 \cdot 5} c_4 = 0}_{\text{all even terms beyond } K=4 \text{ will be zero.}}$ $\therefore y = 3 - 12x^2 + 4x^4$.

23. Find 2 power series solutions of $y'' + (\sin x)y = 0$ about the ordinary pt $x = 0$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

$$0 = y'' + (\sin x)y = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right]\left[c_0 + c_1 x + c_2 x^2 + \dots\right]$$

$$= [2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots]$$

$$+ \left[c_0 x + (c_1)x^2 + \left(c_2 - \frac{c_0}{3!}\right)x^3 + \dots\right]$$

$$= 2c_2 + (c_0 + 6c_3)x + (c_1 + 12c_4)x^2 + \left(20c_5 + c_2 - \frac{c_0}{3!}\right)x^3 + \dots$$

$$\Rightarrow c_2 = 0 \ \& \ c_3 = \frac{-1}{6}c_0 \ \& \ c_4 = \frac{-1}{12}c_1 \ \& \ c_5 = \frac{c_0}{2\cdot 3\cdot 20} - \frac{c_2}{20}$$

$$= \frac{c_0}{120} - 0 = \frac{1}{120}c_0.$$

$$\therefore y = c_0 + c_1 x - \frac{1}{6}c_0 x^3 - \frac{1}{12}c_1 x^4 + \frac{1}{120}c_0 x^5 + \dots$$

$$\Rightarrow y_1 = 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \dots \quad [c_0 = 1, c_1 = 0].$$

$$\& \ y_2 = x - \frac{1}{12}x^4 + \dots \quad [c_0 = 0, c_1 = 1].$$

25. Find a min. rad. of convergence of power series solutions to $(\cos x)y'' + y' + 5y = 0$ about the ordinary pt (a) $x=0$, (b) $x=1$.

$$P(x) = \frac{1}{\cos x} \ \& \ Q(x) = \frac{5}{\cos x}$$

We know $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ is analytic. Since it's

appearing in the denominator, $P(x)$ & $Q(x)$ will fail to be analytic when $\cos x = 0$. i.e. when $x = \frac{\pi}{2} K$, for K an odd integer. $\therefore$ The closest singular pt to 0 & 1 would be $\frac{\pi}{2}$.

$\boxed{a}$ $\sqrt{(\frac{\pi}{2} - 0)^2} = \frac{\pi}{2}$. So, $R = \frac{\pi}{2}$.

$\boxed{b}$ $\sqrt{(\frac{\pi}{2} - 1)^2} = \frac{\pi}{2} - 1$. So, $R = \frac{\pi}{2} - 1$.

26. $\boxed{a}$ Solve $y'' - xy = 1$ about $x = 0$.
$\boxed{b}$ Solve $y'' - 4xy' - 4y = e^x$ about $x = 0$.

$\boxed{a}$ $y'' - xy = 1 \Leftrightarrow \underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{k=n-2} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{k=n+1} = 1$

$\Leftrightarrow \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 1$

$\Leftrightarrow 2c_2 - 1 + \sum_{k=1}^{\infty} \left[(k+2)(k+1) c_{k+2} - c_{k-1} \right] x^k = 0$

$\Rightarrow c_2 = \frac{1}{2}$ & $c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)} \Rightarrow c_3 = \frac{c_0}{3 \cdot 2}$

& $c_4 = \frac{c_1}{4 \cdot 3}$ & $c_5 = \frac{c_2}{5 \cdot 4} = \frac{1}{2 \cdot 5 \cdot 4}$ & $c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{3 \cdot 2 \cdot 6 \cdot 5}$

& $c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{4 \cdot 3 \cdot 7 \cdot 6}$

$\Rightarrow y = c_0 + c_1 x + \frac{1}{2} x^2 + \frac{c_0}{6} x^3 + \frac{c_1}{12} x^4 + \frac{1}{40} x^5 + \cdots$

$= (\frac{1}{2} x^2 + \frac{1}{40} x^5 + \cdots) + c_0 (1 + \frac{1}{6} x^3 + \cdots) + c_1 (x + \frac{1}{12} x^4 + \cdots)$.

[b] $y'' - 4xy' - 4y = e^x \Leftrightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 4\sum_{n=1}^{\infty} n c_n x^n$

$- 4\sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

[Could combine then expand... doesn't matter...].

$\Rightarrow [2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots]$

$- 4[c_1 x + 2c_2 x^2 + 3c_3 x^3 + \dots]$

$- 4[c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots]$

$- [1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots] = 0$

$\Rightarrow 2c_2 - 4c_0 - 1 = 0 \;\&\; 6c_3 - 4c_1 - 4c_1 - 1 = 0$

$\& \; 12c_4 - 8c_2 - 4c_2 - \frac{1}{2} = 0 \;\&\; 20c_5 - 12c_3 - 4c_3 - \frac{1}{6} = 0$

$\Rightarrow c_2 = \frac{1}{2} + 2c_0 \;\&\; c_3 = \frac{1}{6} + \frac{4}{3}c_1$

$\& \; c_4 = \frac{1}{24} + c_2 = \frac{1}{24} + \frac{12}{24} + 2c_0 = \frac{13}{24} + 2c_0$

$\& \; c_5 = \frac{1}{6 \cdot 20} + \frac{16}{20}c_3 = \frac{1}{120} + \frac{4}{5}\left(\frac{1}{6} + \frac{4}{3}c_1\right) = \frac{1}{120} + \frac{4}{30} + \frac{16}{15}c_1$

$\frac{17}{120} + \frac{16}{15}c_1$

$= \frac{17}{120} + \frac{16}{15}c_1$

$\Rightarrow y = c_0 + c_1 x + \left(\frac{1}{2} + 2c_0\right)x^2$

$+ \left(\frac{1}{6} + \frac{4}{3}c_1\right)x^3 + \left(\frac{13}{24} + 2c_0\right)x^4$

$+ \left(\frac{17}{120} + \frac{16}{15}c_1\right)x^5 + \dots$

$= \left(\frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{13}{24}x^4 + \frac{17}{120}x^5 + \dots\right) + c_0\left[1 + 2x^2 + 2x^4 + \dots\right] + c_1$