6.1: Review of Power Series

\# 3, 11, 19, 25, 29, 31

3. Find the interval & radius of convergence for $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$.

Ratio Test: $\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{n+1}}{\frac{2^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{2n}{n+1} \right| = \lim_{n\to\infty} \left| \frac{2}{1+1/n} \right| = 2.$

$\therefore R = 1/2$. Know converges $(-1/2, 1/2)$. Need to check endpoints:

$x = 1/2$: $\sum_{n=1}^{\infty} \frac{2^n}{n} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series $\Rightarrow$ diverges.

$x = -1/2$: $\sum_{n=1}^{\infty} \frac{2^n}{n} (-1)^n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$.

Alternating Series Test: $b_n = \frac{1}{n} > 0$. $b_{n+1} = \frac{1}{n+1}$. $n+1 > n \Rightarrow \frac{1}{n} > \frac{1}{n+1}$
$\Rightarrow b_n \geq b_{n+1} \ \forall n$.

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$. $\therefore$ Converges at $x = -1/2$.

$\therefore [-1/2, 1/2)$ is the interval of convergence.

11. Use an appropriate series to find the Maclaurin series of the given function. Write your answer in summation notation. $e^{-x/2}$.

$$e^{-x/2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2)^n n!} x^n.$$

[since $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$].

19. Find the First 4 nonzero terms of the Maclaurin series of $\sin x \cos x$.

$$\sin x \cos x = \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}\right]\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right]$$

$$= \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right]\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right]$$

$$= x + \left[-\frac{1}{2} - \frac{1}{3!}\right]x^3 + \left[\frac{1}{4!} + \frac{1}{2!3!} + \frac{1}{5!}\right]x^5 + x^7\left[-\frac{1}{6!} - \frac{1}{3!4!} - \frac{1}{2!5!} - \frac{1}{7!}\right]$$

$$= x - \frac{2}{3}x^3 + \frac{2}{15}x^5 - \frac{4}{315}x^7 + \cdots$$

$\frac{1}{24} + \frac{1}{12} + \frac{1}{120} = \frac{16}{120} = \frac{2}{15}$

$-\frac{1}{2} - \frac{1}{6} = -\frac{3}{6} - \frac{1}{6} = -\frac{4}{6} = -\frac{2}{3}$

$-\frac{1}{720} - \frac{1}{144} - \frac{1}{240} - \frac{1}{5040} = -\frac{8}{5040} - \frac{1}{144} - \frac{1}{240} = -\frac{1}{630} - \frac{1}{144} - \frac{1}{240} = -\frac{4}{315}$

#25, 29: Rewrite the given expression using a single power series whose general term involves x^k.

25. $$\sum_{n=1}^{\infty} \underbrace{n c_n x^{n-1}}_{k=n-1} - \sum_{n=0}^{\infty} \underbrace{c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k - \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \left[(k+1)c_{k+1} - c_k\right] x^k.$$

29. $$\sum_{n=2}^{\infty} \underbrace{n(n-1) c_n x^{n-2}}_{k=n-2} - 2\sum_{n=1}^{\infty} \underbrace{n c_n x^n}_{k=n} + \sum_{n=0}^{\infty} \underbrace{c_n x^n}_{k=n}$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - 2\sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} c_k x^k$$

$$= 2c_2 + c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - 2kc_k + c_k\right] x^k$$

31. Verify by direct substitution that $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ is a solution of $y' + 2xy = 0$. [Hint: For x^{2n+1}, let $k = n+1$].

$$y' = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} 2n\, x^{2n-1} = \sum_{n=1}^{\infty} \frac{2(-1)^n}{(n-1)!} x^{2n-1}.$$

$$y' + 2xy = 2\underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} x^{2n-1}}_{n=k} + 2\sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n}{n!} x^{2n+1}}_{k=n+1 \Rightarrow n=k-1 \Rightarrow 2n+1 = 2k-1.}$$

$$= 2\sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} x^{2k-1} + 2\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} x^{2k-1}$$

$$= 2\sum_{k=1}^{\infty} \left[\frac{(-1)^k}{(k-1)!} + \frac{(-1)^{k-1}}{(k-1)!} \right] x^{2k-1} = 2\sum_{k=1}^{\infty} \frac{(-1)^k - (-1)^k}{(k-1)!} x^{2k-1} = 0. \checkmark$$