## 4.7: Cauchy-Euler Eq$^{n}$

\# 1, 5, 15, 21, 27, 40

\# 1, 5, 15, 21, 27: Solve the DE.

1. $x^2y'' - 2y = 0.$

$y = x^m \Rightarrow [m(m-1) - 2]\, x^m = 0$ on $(0, \infty)$

$\Rightarrow m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0 \Rightarrow m = 2$ or $m = -1.$

General solution is $y = c_1 x^2 + c_2 x^{-1}.$

5. $x^2y'' + xy' + 4y = 0.$

$x^m[m(m-1) + m + 4] = 0 \Rightarrow m^2 + 4 = 0 \Rightarrow m = \pm 2i.$

General solution is $y = c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x).$

15. $x^3 y''' - 6y = 0.$

$x^m[m(m-1)(m-2) - 6] = 0 \Rightarrow (m^2 - m)(m-2) - 6 = 0$

$\Rightarrow m^3 - 3m^2 + 2m - 6 = 0.$ By inspection, $m = 3$ works.

$$
\begin{array}{r|l}
 & m^2 + 2 \\
(m-3) & m^3 - 3m^2 + 2m - 6 \\
 & \underline{m^3 - 3m^2} \\
 & \qquad\qquad 2m - 6 \\
 & \qquad\qquad \underline{2m - 6} \\
 & \qquad\qquad\quad 0
\end{array}
$$

$(m-3)(m^2+2) = 0 \Leftrightarrow m = 3$ or $m = \pm\sqrt{2}i$

General solution is

$y = c_1 x^3 + c_2 \cos(\sqrt{2} \ln x) + c_3 \sin(\sqrt{2} \ln x).$

21. $x^2y'' - xy' + y = 2x$.

$x^m[m(m-1) - m + 1] = 0 \Rightarrow m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1.$

$y_c = c_1 \underbrace{x}_{y_1} + c_2 \underbrace{x\ln x}_{y_2}.$ $\quad y'' - x^{-1}y' + x^{-2}y = \underbrace{2x^{-1}}_{f(x)}.$

$W = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x.$

$u = \ln x$
$du = \frac{1}{x}dx$

$W_1 = \begin{vmatrix} 0 & x\ln x \\ 2x^{-1} & \ln x + 1 \end{vmatrix} = -2\ln x. \quad u_1 = \int -\frac{2\ln x}{x}\,dx = -2\int u\,du = -u^2$
$= -(\ln x)^2.$

$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^{-1} \end{vmatrix} = 2. \quad u_2 = \int \frac{2}{x}\,dx = 2\ln x.$

$y = c_1x + c_2x\ln x - x(\ln x)^2 + 2x(\ln x)^2$
$= c_1x + c_2x\ln x + x(\ln x)^2.$

27. $x^2y'' + xy' + y = 0, \; y(1) = 1, \; y'(1) = 2.$

$x^m(m(m-1) + m + 1) = 0 \Rightarrow m^2 + 1 = 0 \Rightarrow m = \pm i.$

$y = c_1\cos(\ln x) + c_2\sin(\ln x). \quad y(1) = 1 \Rightarrow 1 = c_1.$

$y' = -c_1\sin(\ln x)\frac{1}{x} + c_2\cos(\ln x)\frac{1}{x}. \quad y'(1) = 2 \Rightarrow 2 = c_2.$

$\therefore y = \cos(\ln x) + 2\sin(\ln x).$

40. Use $y = (x - x_0)^m$ to solve. $(x-1)^2 y'' - (x-1)y' + 5y = 0$.

Let $y = (x-1)^m$. Then

$(x-1)^m [m(m-1) - m + 5] = 0 \Rightarrow m^2 - 2m + 5 = 0$

$\Rightarrow m = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$. $\alpha = 1$, $\beta = 2$.

General solution is:

$y = (x-1)\,[c_1 \cos(2\ln(x-1)) + c_2 \sin(2\ln(x-1))]$.