4.6: Variation of Parameters

1, 11, 17, 23, 25, ~~29~~, 32

#1, 11, 17: Solve by variation of parameters.

1. $y'' + y = \sec x.$

$m^2 + 1 = 0 \Leftrightarrow m = \pm i.$ $\quad y_c = c_1 \underbrace{\cos x}_{y_1} + c_2 \underbrace{\sin x}_{y_2}.$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1.$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x \sec x = -\tan x.$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1.$$

$$u_1 = \int -\tan x \, dx = \ln|\cos x|.$$

$$u_2 = \int dx = x.$$

$$y_p = \ln|\cos x| \cos x + x \sin x.$$

$$y = c_1 \cos x + c_2 \sin x + \cos x \ln|\cos x| + x \sin x.$$

11. $y'' + 3y' + 2y = \frac{1}{1+e^x}.$

$m^2 + 3m + 2 = 0 \Leftrightarrow (m+2)(m+1) = 0 \Leftrightarrow m = -2$ or $m = -1.$

$y_c = c_1 \underbrace{e^{-2x}}_{y_1} + c_2 \underbrace{e^{-x}}_{y_2}.$ $\quad W(y_1, y_2) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix} = -e^{-3x} + 2e^{-3x} = e^{-3x}.$

$$W_1 = \begin{vmatrix} 0 & e^{-x} \\ \frac{1}{1+e^x} & -e^{-x} \end{vmatrix} = -\frac{e^{-x}}{1+e^x}.$$

$$\frac{e^{2x}+e^x-e^x}{1+e^x}$$

$$W_2 = \begin{vmatrix} e^{-2x} & 0 \\ -2e^{-2x} & \frac{1}{1+e^x} \end{vmatrix} = \frac{e^{-2x}}{1+e^x}.$$

$$= \frac{e^x(e^x+1)}{1+e^x} - \frac{e^x}{1+e^x}$$

$$u_1 = \int -\frac{e^{-x}}{(1+e^x)e^{-3x}}\,dx = \int \frac{-e^{2x}}{1+e^x}\,dx = \int -e^x + \frac{e^x}{1+e^x}\,dx$$

$u = 1+e^x$

$du = e^x\,dx$

$$= -e^x + \int \frac{1}{u}\,du = -e^x + \ln(1+e^x).$$

$$u_2 = \int \frac{e^x}{1+e^x} = \ln(1+e^x).$$

$$y = c_1e^{-2x} + c_2e^{-x} - e^{-x} + e^{-2x}\ln(1+e^x) + e^{-x}\ln(1+e^x)$$

$$= c_1e^{-2x} + c_2e^{-x} + (e^{-2x} + e^{-x})\ln(1+e^x).$$

17. $3y'' - 6y' + 6y = e^x \sec x.$ $f(x) = \frac{e^x \sec x}{3}.$

$$3m^2 - 6m + 6 = 0 \Leftrightarrow m^2 - 2m + 2 = 0 \Leftrightarrow m = \frac{2 \pm \sqrt{4-8}}{2}$$

$$y_c = e^x(c_1\cos x + c_2 \sin x). \qquad = 1 \pm i.$$

$$W = \begin{vmatrix} e^x\cos x & e^x \sin x \\ e^x\cos x - e^x \sin x & e^x\sin x + e^x \cos x \end{vmatrix} = e^{2x}.$$

$$W_1 = \begin{vmatrix} 0 & e^x \sin x \\ \frac{1}{3}e^x \sec x & e^x\sin x + e^x\cos x \end{vmatrix} = -\frac{1}{3}e^{2x}\tan x.$$

$$W_2 = \begin{vmatrix} e^x \cos x & 0 \\ e^x\cos x - e^x \sin x & \frac{1}{3}e^x \sec x \end{vmatrix} = \frac{1}{3}e^{2x}.$$

$u_1 = \int -\frac{1}{3}\tan x\,dx = \frac{1}{3}\ln|\cos x|.$

$u_2 = \int \frac{1}{3}\,dx = \frac{1}{3}x\,dx.$

$y = c_1 e^x \cos x + c_2 e^x \sin x + \frac{1}{3}\ln|\cos x|\,e^x \cos x + \frac{x}{3}e^x \sin x$

23. Find the general solution of $x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{3/2}$, where $y_1 = x^{-1/2}\cos x$ & $y_2 = x^{-1/2}\sin x$ are linearly independent solutions of the assoc. homog. DE on $(0,\infty)$.

$y_c = c_1 \underbrace{x^{-1/2}\cos x}_{y_1} + c_2 \underbrace{x^{-1/2}\sin x}_{y_2}.$

$W = \begin{vmatrix} x^{-1/2}\cos x & x^{-1/2}\sin x \\ -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x & -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x \end{vmatrix}$

$= x^{-1}.$ $\qquad y'' + \frac{1}{x}y' + \frac{(x^2 - \frac{1}{4})}{x^2}y = \underbrace{x^{-1/2}}_{f(x)}.$

$W_1 = \begin{vmatrix} 0 & x^{-1/2}\sin x \\ x^{-1/2} & \sim \end{vmatrix} = -x^{-1}\sin x.$

$W_2 = \begin{vmatrix} x^{-1/2}\cos x & 0 \\ \sim & x^{-1/2} \end{vmatrix} = x^{-1}\cos x.$

$u_1 = \int -\sin x\,dx = \cos x.$

$u_2 = \int \cos x\,dx = \sin x.$

$y = c_1 x^{-1/2}\cos x + c_2 x^{-1/2}\sin x + x^{-1/2}.$

25. Solve $y''' + y' = \tan x$ by variation of parameters.

$$m^3 + m = 0 \iff m(m^2+1) = 0 \iff m = 0 \text{ or } m = \pm i.$$

$$y_c = \underbrace{c_1}_{y_1} + c_2 \underbrace{\cos x}_{y_2} + c_3 \underbrace{\sin x}_{y_3}.$$

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = 1.$$

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x.$$

$$W_2 = -\tan x \cos x = -\sin x.$$

$$W_3 = \tan x(-\sin x).$$

$$u_1 = \int \tan x \, dx = -\ln|\cos x|.$$

$$u_2 = \int -\sin x \, dx = \cos x.$$

$$u_3 = \int \frac{-\sin^2 x}{\cos x} dx = \int \frac{-1+\cos^2 x}{\cos x} dx = \int -\sec x + \cos x \, dx$$

$$= -\ln|\sec x + \tan x| + \sin x.$$

$$y = c_1 + c_2 \cos x + c_3 \sin x - \ln|\cos x| + 1 - \sin x \ln|\sec x + \tan x|$$

$$= c_1 + c_2 \cos x + c_3 \sin x - \ln|\cos x| - \sin x \ln|\sec x + \tan x|.$$

29. Discuss how methods of undetermined coef. & variation of parameters can be combined to solve $3y''-6y'+30y = 15\sin x + e^x \tan(3x)$.

$3m^2-6m+30=0 \iff m^2-2m+10=0 \iff m=\dfrac{2\pm\sqrt{4-40}}{2}$

$y_c = e^x[c_1\cos(3x)+c_2\sin(3x)]$. $= 1\pm 3i$. $\alpha=1, \beta=3$.

Use undetermined coef. to solve $3y''-6y'+30y=15\sin x$. $\alpha=0$ $\beta=1$

$(D^2+1)(3D^2-6D+30)\,y=0$

$m=\pm i$ or $\pm 1+3i$.

So, $y_p = c_3\cos x + c_4 \sin x$.
$y_p' = -c_3\sin x + c_4\cos x$.
$y_p'' = -c_3\cos x - c_4\sin x$.

$15\sin x = -3c_3\cos x - 3c_4\sin x + 6c_3\sin x - 6c_4\cos x + 30c_3\cos x + 30c_4\sin x$

$\iff 27c_3-6c_4=0$ & $27c_4+6c_3=15$

$\Rightarrow c_4 = \frac{27}{6}c_3 = \frac{9}{2}c_3$. $\frac{255}{2}c_3 = 15 \Rightarrow c_3 = \frac{2}{17} \Rightarrow c_4 = \frac{9}{17}$.

$\therefore y_{p_1} = \frac{2}{17}\cos x + \frac{9}{17}\sin x$.

Use variation of par. to solve $3y''-6y'+30y = e^x\tan(3x)$.

$f(x) = \dfrac{e^x\tan(3x)}{3}$

$W = \begin{vmatrix} e^x\cos(3x) & e^x\sin(3x) \\ e^x\cos(3x)-3e^x\sin(3x) & e^x\sin(3x)+3e^x\cos(3x) \end{vmatrix} = 3e^{2x}$.

$W_1 = \begin{vmatrix} 0 & e^x\sin(3x) \\ \frac{1}{3}e^x\tan(3x) & \sim \end{vmatrix} = -\frac{1}{3}e^{2x}\sin(3x)\tan(3x)$.

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$$W_2 = \begin{vmatrix} e^x\cos(3x) & 0 \\ \sim & \frac{1}{3}e^x\tan(3x) \end{vmatrix} = \frac{1}{3}e^{2x}\sin(3x).$$

$$u_1 = \frac{1}{27}\sin x - \frac{1}{27}\ln|\sec x + \tan x|.$$

$$u_2 = \frac{1}{9}\int \sin(3x)\,dx = -\frac{1}{9}\cos(3x).$$

$$y_{p_2} = e^x\cos(3x)\left[\frac{1}{27}\sin 3x - \frac{1}{27}\ln|\sec 3x + \tan 3x|\right. - \frac{1}{9}e^x\sin(3x)\cos(3x).$$

$$\therefore\ y = e^x\left(c_1\cos(3x) + c_2\sin(3x)\right) + y_{p_1} + y_{p_2}.$$

32. Find the general solution of $x^4y'' + x^3y' - 4x^2y = 1$ given $y_1 = x^2$ solution to assoc. homog. eq'n.

By reduction of order with $y'' + \underbrace{\frac{1}{x}}_{P(x)}y' \underbrace{- \frac{4}{x^2}}_{Q(x)}y = \underbrace{\frac{1}{x^4}}_{f(x)}$

we have $y_2 = x^2\int \frac{e^{-\ln x}}{x^4}\,dx$

$$= x^2\int x^{-5}\,dx = x^2\left[-\frac{1}{4}x^{-4}\right] = -\frac{1}{4}x^{-2}.$$

So, $y_c = c_1\underbrace{x^2}_{y_1} + c_2\underbrace{x^{-2}}_{y_2}$.

$$W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1}.$$

$$W_1 = \begin{vmatrix} 0 & x^{-2} \\ x^{-4} & -2x^{-3} \end{vmatrix} = -x^{-6}. \quad u_1 = \frac{1}{4}\int x^{-5}\,dx = -\frac{1}{16}x^{-4}$$

$$W_2 = \begin{vmatrix} x^2 & 0 \\ 2x & x^{-4} \end{vmatrix} = x^{-2}. \quad u_2 = -\frac{1}{4}\int x^{-1}\,dx = -\frac{1}{4}\ln x.$$

So, $y_p = -\frac{1}{16}x^{-2} - \frac{1}{4}x^{-2}\ln x$.

$\therefore$ The general solution is:

$y = c_1x^2 + c_2x^{-2} - \frac{1}{16}x^{-2} - \frac{1}{4}x^{-2}\ln x$.