4.5: Undetermined Coefficients - Annihilator Approach

#5, 21, 29, 35, 39, 53, 69

5. Write the DE $y''' + 10y'' + 25y' = e^x$ in the form $L(y) = g(x)$.

$(D^3 + 10D^2 + 25D)y = e^x$

$\Leftrightarrow D(D^2 + 10D + 25)y = e^x \Leftrightarrow D(D+5)^2 y = e^x.$

21. Find the L that annihilates $13x + 9x^2 - \sin(4x)$.

$L = D^3(D^2 + 16).$ $\qquad \alpha = 0, \beta = 4$

29. Find linearly independent functions that are annihilated by $(D-6)(2D+3)$.

$(D-6)$ annihilates e^{6x} & $(2D+3)$ annihilates $e^{-3x/2}$.

$c_1 e^{6x} + c_2 e^{-3x/2} = 0 \Leftrightarrow c_1 = c_2 = 0 \Rightarrow e^{6x}$ & $e^{-3x/2}$ are lin. independent.
$\therefore e^{6x}$ & $e^{-3x/2}$.

#35, 39, 53, 69: Solve by undetermined coefficients.

35. $y'' - 9y = 54$

$m^2 - 9 = 0 \Leftrightarrow (m-3)(m+3) = 0 \Leftrightarrow m = 3$ or $m = -3$.
$y_c = \underline{c_1 e^{3x}} + \underline{c_2 e^{-3x}}.$

$D(54) = 0.$ $\qquad D(D^2 - 9)y = D(54) = 0.$

$m(m^2 - 9) \Leftrightarrow m = 0$ or $m = 3$ or $m = -3$.
$y = c_1 + \underline{c_2 e^{3x}} + \underline{c_3 e^{-3x}}.$ $\qquad y_p = c_1.$

$-9c_1 = 54 \Rightarrow c_1 = -6. \quad y_p = -6.$

$\therefore \; y = c_1 e^{3x} + c_2 e^{-3x} - 6.$

39. $y'' + 4y' + 4y = 2x + 6.$

$m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m = -2.$

$y_c = \underline{c_1 e^{-2x}} + \underline{c_2 x e^{-2x}}.$

$D^2 (D^2 + 4D + 4) y = 0. \quad m = 0 \quad m = -2.$

$y = c_1 + c_2 x + \underline{c_3 e^{-2x}} + \underline{c_4 x e^{-2x}}.$

$y_p = c_1 + c_2 x.$
$y_p' = c_2.$

$$\left.\begin{aligned} 4c_2 + 4c_1 + 4c_2 x &= 2x + 6 \\ (4c_2)x + (4c_1 + 4c_2) &= 2x + 6 \end{aligned}\right\} \quad \begin{aligned} 4c_2 &= 2 \\ c_2 &= 1/2 \end{aligned} \quad \begin{aligned} 4c_1 + 2 &= 6 \\ 4c_1 &= 4 \\ c_1 &= 1. \end{aligned}$$

$y_p = 1 + x/2.$

$y = c_1 e^{-2x} + c_2 x e^{-2x} + \frac{x}{2} + 1.$

53. $y'' - 2y' + 5y = e^x \sin x. \; \rightarrow \alpha = 1, \beta = 1$

$m^2 - 2m + 5 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i. \quad \alpha = 1, \beta = 2.$

$y_c = \underline{e^x [c_1 \cos(2x) + c_2 \sin(2x)]}.$

$(D^2 - 2D + 2)(D^2 - 2D + 5) y = 0 \qquad m = \frac{2 \pm \sqrt{4-8}}{2}$

$(m^2 - 2m + 2)(m^2 - 2m + 5) = 0 \qquad = 1 \pm i. \quad \alpha = \beta = 1.$

$y = e^x (c_1 \cos x + c_2 \sin x) + \underline{e^x (c_1 \cos(2x) + c_2 \sin(2x))}$

$$y_p = e^x(c_1\cos x + c_2 \sin x).$$

$$y_p' = e^x(c_1\cos x + c_2 \sin x) + e^x(-c_1\sin x + c_2\cos x)$$
$$= e^x((c_2-c_1)\sin x + (c_1+c_2)\cos x).$$

$$y_p'' = e^x((c_2-c_1)\sin x + (c_1+c_2)\cos x)$$
$$+ e^x((c_2-c_1)\cos x - (c_1+c_2)\sin x)$$
$$= e^x[-2c_1\sin x + 2c_2\cos x].$$

$$e^x[-2c_1\sin x + 2c_2\cos x - 2(c_2-c_1)\sin x - 2(c_1+c_2)\cos x$$
$$+ 5c_1\cos x + 5c_2\sin x] = e^x\sin x$$

$$\Rightarrow 3c_2e^x\sin x + 3c_1\cos x\, e^x = e^x\sin x \Rightarrow c_1 = 0 \;\&\; c_2 = 1/3.$$

$$y_p = \tfrac{1}{3}e^x\sin x.$$

$$y = e^x[c_1\cos(2x) + c_2\sin(2x)] + \tfrac{1}{3}e^x\sin x.$$

69. $y'' + y = \underbrace{8\cos(2x)}_{\alpha=0,\beta=2} - \underbrace{4\sin x}_{\alpha=0,\beta=1},\ y(\frac{\pi}{2}) = -1,\ y'(\frac{\pi}{2}) = 0.$

$m^2+1 = 0 \Leftrightarrow m = \pm i.$

$y_c = c_1\cos x + c_2\sin x.$

$(D^2+4)(D^2+1)(D^2+1) = 0. \quad m = \underbrace{\pm 2i}_{\alpha=0,\beta=2}\ \&\ \underbrace{\pm i}_{\alpha=0,\beta=1}$

$$y_p = c_1\cos(2x) + c_2\sin(2x) + c_3x\cos x + c_4x\sin x$$
$$y_p' = -2c_1\sin(2x) + 2c_2\cos(2x) - c_3x\sin x + c_4x\cos x + c_3\cos x + c_4\sin x$$
$$y_p'' = -4c_1\cos(2x) - 4c_2\sin(2x) - c_3x\cos x - c_4x\sin x - c_3\sin x + c_4\cos x - c_3\sin x + c_4\cos x$$

$$-3c_1\cos(2x) - 3c_2\sin(2x) - 2c_3\sin x + 2c_4\cos x = 8\cos(2x) - 4\sin x$$

$$\Rightarrow c_1 = -8/3,\ c_2 = 0,\ c_3 = 2,\ c_4 = 0. \quad y_p = -\tfrac{8}{3}\cos(2x) + 2x\cos x.$$
$$y = c_1\cos x + c_2\sin x - 8/3\cos(2x) + 2x\cos x.$$

$y(\frac{\pi}{2}) = -1 \Rightarrow -1 = \frac{-8}{3}(-1) + c_2 \Rightarrow c_2 = \frac{-3}{3} - \frac{8}{3} = \frac{-11}{3}.$

$y' = -(2x + c_1)\sin x + c_2 \cos x + \frac{16}{3}\sin(2x).$

$y'(\frac{\pi}{2}) = 0 \Rightarrow 0 = -\frac{2\pi}{2} - c_1 \ni c_1 = -\pi.$

$\therefore y = (-\pi + 2x)\cos x - \frac{11}{3}\sin x - \frac{8}{3}\cos(2x).$