4.2: Reduction of order

3, 11, 17

3, 11: The indicated function $y_1(x)$ is a solution of the given DE. Use reduction of order to find a second solution $y_2(x)$.

3. $y'' + \underbrace{16}_{Q(x)}y = 0$; $y_1 = \cos(4x)$.

Recall: $y_2 = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1^2(x)} dx$, where $y'' + P(x)y' + Q(x)y = 0$.

Here $P(x) = 0$. So, $y_2 = \cos(4x) \int \frac{1}{\cos^2(4x)} dx$

$u = 4x$
$du = 4dx$
$\frac{1}{4}du = dx$

$= \cos(4x) \int \sec^2(4x)\, dx = \frac{1}{4}\cos(4x)[\tan(4x) + c]$.

So, a second solution would be $y_2 = \frac{1}{4}\cos(4x)\tan(4x)$
$= \frac{1}{4}\sin(4x)$.

[Any c value would be fine. Also $\sin(4x)$ works too].

11. $xy'' + y' = 0$; $y_1 = \ln x$.

$y'' + \underbrace{\frac{1}{x}}_{P(x)} y' = 0$. $y_2 = \ln x \int \frac{e^{-\int \frac{1}{x}dx}}{(\ln x)^2} dx$

$u = \ln x$
$du = \frac{1}{x}dx$

$= \ln x \int \frac{e^{\ln(x^{-1})}}{(\ln x)^2} dx = \ln x \int \frac{x^{-1}}{(\ln x)^2} dx = \ln x \int u^{-2} du$

$= [-u^{-1} + c]\ln x = \left[\frac{-1}{\ln x} + c\right]\ln x = -1 + \frac{c}{\ln x}$.

So, a possible second solution would be $y_2 = 1$.

[Again, any value of c + any scalar multiple is fine].

17. Here $y_1(x)$ is a solution of the associated homog. eq'n. Use reduction of order to find a second solution $y_2(x)$ of the homog. eq'n & a particular solution of the nonhomog. eq'n.

$y'' - 4y = 2$; $y_1 = e^{-2x}$.

$y'' \underbrace{-4}_{Q(x)} y = 0.$ $\quad y_2 = y_1(x) \int \frac{e^{-\int P(x)dx}}{(y_1(x))^2}\,dx$

$= e^{-2x} \int \frac{1}{e^{-4x}}\,dx = e^{-2x} \int e^{4x}\,dx = \frac{1}{4} e^{-2x} [e^{4x} + c].$

So, a solution would be $y_2(x) = e^{2x}$.

By inspection $y_p = -\frac{1}{2}$ is a particular solution:
$0 - 4(-\frac{1}{2}) = 2.$ ✓

The general solution of this eq'n is
$y = c_1 e^{-2x} + c_2 e^{2x} - \frac{1}{2}.$