4.1: Preliminary Theory - Linear Eq'ns

5, 6, 9, 15, 21, 29, 31

5. Given that $y = c_1 + c_2 x^2$ is a 2-parameter family of solutions of $xy'' - y' = 0$ on $(-\infty, \infty)$, show that constants c_1 & c_2 can't be found so that a member of the family satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$. Explain why this doesn't violate Theorem 4.1.1.

$y' = 2c_2 x$. $y'(0) = 1 \Rightarrow 1 = 0$. Not possible!
$\therefore$ No member of the family satisfies these initial conditions. This doesn't contradict Theorem 4.1.1, b/c $a_2(x) = x$ is zero at $x = 0$ & $0 \in (-\infty, \infty)$, so assumptions of Theorem not satisfied.

6. Find 2 members of the family of solutions in #5 that satisfy $y(0) = 0$ & $y'(0) = 0$.

$y(0) = 0 \Rightarrow 0 = c_1$. $\therefore$ Solutions of the form
$y'(0) = 0 \Rightarrow 0 = 0$. $y = c_2 x^2$ satisfy the DE & these initial conditions.

So, 2 possible members would be $\underbrace{y = x^2 \text{ \& } y = 8x^2}_{\text{[Picking any value for } c_2 \text{ would be fine].}}$

9. Find an interval centered about $x = 0$ for which $(x-2)y'' + 3y = x$, $y(0) = 0$, $y'(0) = 1$ has a unique solution.

$(x-2)$, 3, & x cont. everywhere. Need an interval when $x - 2 \neq 0$, so $x = 2$ can't be in it. $\therefore$ $(-\infty, 2)$ would work.
i.e. $\exists!$ solution to this IVP in $(-\infty, 2)$.

15, 21: Determine whether the given set of functions is linearly independent on $(-\infty, \infty)$.

15. $f_1(x) = x,\ f_2(x) = x^2,\ f_3(x) = 4x - 3x^2.$

Here $4f_1 - 3f_2 - f_3 = 0 \Rightarrow$ linearly dependent.
(i.e. $4x - 3x^2 - (4x - 3x^2) = 0$)

21. $f_1(x) = 1 + x,\ f_2(x) = x,\ f_3(x) = x^2.$

Suppose $c_1 f_1 + c_2 f_2 + c_3 f_3 = 0$

$\Leftrightarrow c_1(1+x) + c_2 x + c_3 x^2 = 0$
$\Leftrightarrow c_3 x^2 + (c_1 + c_2)x + c_1 = 0 \Rightarrow c_1 = 0,\ c_3 = 0,\ c_1 + c_2 = 0$
$\Leftrightarrow c_1 = c_2 = c_3 = 0 \Rightarrow$ linearly independent.

29. Verify $x^3 y''' + 6x^2 y'' + 4xy' - 4y = 0$; $x, x^{-2}, x^{-2}\ln x$ form a fundamental set of solutions on $(0, \infty)$. Form the general solution.

These 3 functions all satisfy the DE:

$\underline{x}$: $0 + 0 + 4x - 4x = 0.$ ✓

$-2x^{-3},\ 6x^{-4},\ -24x^{-5}$

$\underline{x^{-2}}$: $x^3(-24x^{-5}) + 6x^2(6x^{-4}) + 4x(-2x^{-3}) - 4x^{-2}$
$= -24x^{-2} + 36x^{-2} - 8x^{-2} - 4x^{-2} = 0.$ ✓

$y' = -2x^{-3}\ln x + x^{-3} = -x^{-3}(2\ln x - 1)$

$y'' = 3x^{-4}(2\ln x - 1) - 2x^{-4}$

$y''' = -12x^{-5}(2\ln x - 1) + 6x^{-5} + 8x^{-5}$

$\underline{x^{-2}\ln x}$: $x^3(26x^{-5} - 24x^{-5}\ln x) + 6x^2(-5x^{-4} + 6x^{-4}\ln x)$
$+ 4x(x^{-3} - 2x^{-3}\ln x) - 4x^{-2}\ln x$
$= (26 - 30 + 4)x^{-2} + (-24 + 36 - 8 - 4)x^{-2}\ln x = 0.$ ✓

Also, all 3 possess at least 2 derivatives on $(0, \infty)$, so can use Theorem 4.1.3:

$$W(x, x^{-2}, x^{-2}\ln x) = \begin{vmatrix} x & x^{-2} & x^{-2}\ln x \\ 1 & -2x^{-3} & x^{-3} - 2x^{-3}\ln x \\ 0 & 6x^{-4} & -5x^{-4} + 6x^{-4}\ln x \end{vmatrix}$$

$$= x \begin{vmatrix} -2x^{-3} & x^{-3} - 2x^{-3}\ln x \\ 6x^{-4} & -5x^{-4} + 6x^{-4}\ln x \end{vmatrix} - \begin{vmatrix} x^{-2} & x^{-2}\ln x \\ 6x^{-4} & -5x^{-4} + 6x^{-4}\ln x \end{vmatrix}$$

$$= 10x^{-6} - 12\cancel{x^{-6}\ln x} - 6x^{-6} + 12\cancel{x^{-6}\ln x} - [-5x^{-6} + 6\cancel{x^{-6}\ln x} - 6\cancel{x^{-6}\ln x}]$$

$$= 4x^{-6} + 5x^{-6} = \frac{9}{x^6} \neq 0 \text{ for } x \in (0, \infty).$$

$\therefore \{x, x^{-2}, x^{-2}\ln x\}$ form a fundamental set of solutions.

The general solution on $(0, \infty)$ is:

$$y = c_1 x + c_2 x^{-2} + c_3 x^{-2}\ln x.$$

31. Verify that $y'' - 7y' + 10y = 24e^x$; $y = c_1 e^{2x} + c_2 e^{5x} + 6e^x$, $(-\infty, \infty)$ gives a general solution to this non homg. DE.

We need to check to see that $\{e^{2x}, e^{5x}\}$ form a fundamental solution set for $y'' - 7y' + 10y = 0$, & that $6e^x$ is a particular solution to $y'' - 7y' + 10y = 24e^x$.

$\underline{6e^x}$: $6e^x - 42e^x + 60e^x = 24e^x \Rightarrow 6e^x$ particular solution.

$\underline{e^{5x}}$: $25e^{5x} - 35e^{5x} + 10e^{5x} = 0$. ✓

$\underline{e^{2x}}$: $4e^{2x} - 14e^{2x} + 10e^{2x} = 0$. ✓

$$\begin{vmatrix} e^{2x} & e^{5x} \\ 2e^{2x} & 5e^{5x} \end{vmatrix} = 5e^{7x} - 2e^{7x} = 3e^{7x} \neq 0 \text{ for } x \in (-\infty, \infty)$$

$\Rightarrow \{e^{5x}, e^{2x}\}$ fundamental solution set

$\Rightarrow y = c_1 e^{2x} + c_2 e^{5x} + 6e^x$ general solution to nonhomg. eq'n.