2.5: Solutions by Substitution:

#1, 13, 17, 27

1, 13, 17, 27: Solve the DE by using an appropriate substitution.

1. $\underbrace{(x-y)}_{M}\,dx + \underbrace{x}_{N}\,dy = 0.$

M & N are both homog. of deg. 1.

$$\left.\begin{aligned} y &= ux \\ dy &= x\,du + u\,dx. \end{aligned}\right\} \Rightarrow (x-ux)\,dx + x(x\,du + u\,dx) = 0$$

$$\Rightarrow x\,dx + x^2\,du = 0$$

$$\Rightarrow \int \frac{1}{x}\,dx = -\int du$$

$$\Rightarrow \ln x = -u + c \Rightarrow \ln x = \frac{-y}{x} + c \Rightarrow x\ln x = -y + cx$$

$u = y/x$

$$\Rightarrow y = -x\ln x + cx, \text{ on } (0,\infty).$$

13. $\underbrace{(x + y e^{y/x})}_{M}\,dx \underbrace{- x e^{y/x}}_{N}\,dy = 0, \quad y(1) = 0.$

M & N both homog. of deg. 1. Indeed, $M(tx, ty)$

$= (tx + ty e^{ty/tx}) = t\,M(x,y)$, & $N(tx,ty) = -t x e^{ty/tx} = tN(x,y)$.

$$\left.\begin{aligned} y &= ux \\ dy &= u\,dx + x\,du \end{aligned}\right\} \Rightarrow (x + uxe^u)\,dx - xe^u(u\,dx + x\,du) = 0$$

$$\Rightarrow x\,dx - e^u x^2\,du = 0 \Rightarrow \int \frac{1}{x}\,dx = \int e^u\,du$$

$\Rightarrow \ln x = e^{u} + c$

$\Rightarrow \ln x = e^{y/x} + c.$

$y(1) = 0 \Rightarrow \ln(1) = e^{0} + c$

$\Rightarrow 0 - 1 = c \Rightarrow c = -1.$

$\therefore \underbrace{\ln|x| = e^{y/x} - 1}_{\text{implicit solution}}$

17. $\frac{dy}{dx} = y(xy^3 - 1).$

$\frac{dy}{dx} + y = xy^4.$ Here $n = 4.$ $u = y^{1-4} = y^{-3}$

$\frac{du}{dx} = -3y^{-4}\frac{dy}{dx}$

$y^{-4}\frac{dy}{dx} + y^{-3} = x$

$-\frac{1}{3}\frac{du}{dx} = y^{-4}\frac{dy}{dx}$

$-\frac{1}{3}\frac{du}{dx} + u = x$

$\frac{du}{dx} \underbrace{-3u}_{P(x)} = \underbrace{-3x}_{F(x)}$

$\int P(x)\,dx = \int -3\,dx = -3x.$

$w = x$
$dw = dx$
$v = -\frac{1}{3}e^{-3x}$
$dv = e^{-3x}$

$u = e^{3x}\left[\int e^{-3x}(-3x)\,dx\right]$

$= -3e^{3x}\left[-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\,dx\right]$

$= -e^{3x}\left[-xe^{-3x} - \frac{1}{3}e^{-3x} + c\right]$

$\Rightarrow \boxed{y^{-3} = x + \frac{1}{3} + ce^{3x}}$

implicit solution

27. $\frac{dy}{dx} = 2 + \sqrt{y-2x+3}$.

Here we'll use "Reduction to Separation of Variables", since we can let $F(u) = 2 + \sqrt{u}$, where $u = y - 2x + 3$.
[i.e.] $A = -2,\ B = 1,\ C = 3$].

$$\frac{du}{dx} = -2 + \frac{dy}{dx}.$$

$$\frac{dy}{dx} = \frac{du}{dx} + 2.$$

$$\frac{du}{dx} + 2 = 2 + u^{1/2}$$

$$\int u^{-1/2}\,du = \int dx$$

$$2u^{1/2} = x + c$$

$$\sqrt{y-2x+3} = \tfrac{1}{2}x + c$$

$$y - 2x + 3 = \left(\tfrac{1}{2}x + c\right)^2$$

$$y = \left(\tfrac{1}{2}x + c\right)^2 + 2x - 3 \text{ on } (-\infty, \infty).$$