2.4: Exact Eq^n's:

#1, 7, 25, 31, ~~37~~, 39, 40, 44

1, 7: Determine whether the DE is exact. If it is, solve it.

Recall: • A First-order DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is an exact eq^n $\Leftrightarrow$ $M(x,y)\,dx + N(x,y)\,dy = \nabla F$ for some function $F(x,y)$. If $M(x,y)$ & $N(x,y)$ are C^1 in a rectangular region $R = [a,b] \times [c,d]$, then $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact $\Leftrightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

• To solve an exact DE, we want to find the F s.t. $\nabla F = M(x,y)\,dx + N(x,y)\,dy$. Then an implicit solution is $F(x,y) = C$.

1. $\underbrace{(2x-1)}_{M(x,y)}dx + \underbrace{(3y+7)}_{N(x,y)}dy = 0.$

M & N are both C^1 in any rectangular region in $\mathbb{R}^2$ since they're polynomials.

$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} \Rightarrow$ exact.

$\nabla F = (M, N) \Rightarrow \frac{\partial F}{\partial x} = M$ & $\frac{\partial F}{\partial y} = N.$

$\frac{\partial F}{\partial x} = M \Rightarrow F = \int M\,dx \Rightarrow F = \int (2x-1)\,dx$

$= x^2 - x + g(y).$ $\frac{\partial F}{\partial y} = N \Rightarrow 0 + \frac{\partial g}{\partial y} = 3y+7$

$\Rightarrow g = \int 3y+7\,dy = \frac{3}{2}y^2 + 7y + C.$

$\therefore F = x^2 - x + \frac{3}{2}y^2 + 7y + C.$

$\therefore$ The implicit solution is $x^2 - x + \frac{3}{2}y^2 + 7y = c$.

7. $(x^2 - y^2)dx + (x^2 - 2xy)dy = 0$.

M & N polynomials $\Rightarrow C^1$ in any rectangular region.

$\frac{\partial M}{\partial y} = -2y$. $\frac{\partial N}{\partial x} = 2x - 2y$. $\therefore$ Not exact.

Not same

25. Solve the IVP $(\underbrace{y^2\cos x - 3x^2y - 2x}_{M})dx + (\underbrace{2y\sin x - x^3 + \ln y}_{N})dy = 0$, $y(0) = e$.

Not hard to see that we can find a rect. regions where M & N are C^1.

$\frac{\partial M}{\partial y} = 2y\cos x - 3x^2 = \frac{\partial N}{\partial x} \Rightarrow$ exact.

$DF = (M, N) \Rightarrow \frac{\partial F}{\partial x} = M$ & $\frac{\partial F}{\partial y} = N$.

$\frac{\partial F}{\partial x} = M \Rightarrow F = \int y^2\cos x - 3x^2y - 2x \, dx$

$\Rightarrow F = y^2\sin x - x^3y - x^2 + g(y)$.

$\frac{\partial F}{\partial y} = N \Rightarrow 2y\sin x - \cancel{x^3} + \frac{\partial g}{\partial y} = 2y\sin x - \cancel{x^3} + \ln y$

$\Rightarrow g = \int \ln y \, dy = y\ln y - y + c$.

$\therefore F = y^2\sin x - x^3y - x^2 + y\ln y - y + c$.

$y^2\sin x - x^3y - x^2 + y\ln y - y = C$. $y(0) = e \Rightarrow e\ln e - e = C$
$\Rightarrow C = 0$.

$\therefore$ Implicit solution is $y^2\sin x - x^3y - x^2 + y\ln y - y = 0$.

31. Solve $(2y^2+3x)dx + 2xy\,dy = 0$ by finding an appropriate integrating factor.

Recall: If $\frac{M_y - N_x}{N}$ is a function of x alone, then an integrating factor is $\mu(x) = e^{\int \frac{M_y - N_x}{N}dx}$.

Similarly, if $\frac{N_x - M_y}{M}$ function of y alone, then an integrating factor is $\mu(y) = e^{\int \frac{N_x - M_y}{M}dy}$.

Notice that the original eq'n not exact since $\frac{\partial M}{\partial y} = 4y$ & $\frac{\partial N}{\partial x} = 2y$.

$$\frac{M_y - N_x}{N} = \frac{4y - 2y}{2xy} = \frac{2y}{2xy} = 1/x \text{ depends on } x \text{ alone.}$$

So, $\mu(x) = e^{\int \frac{M_y - N_x}{N}dx} = e^{\int 1/x\,dx} = e^{\ln(x)} = x$ is an integrating factor.

Indeed, $\overbrace{(2xy^2 + 3x^2)}^{\tilde{M}}dx + \overbrace{2x^2y\,dx}^{\tilde{N}} = 0$ is exact, since $\frac{\partial \tilde{M}}{\partial y} = 4xy = \frac{\partial \tilde{N}}{\partial x}$. $\nabla f = (\tilde{M}, \tilde{N}) \Rightarrow \frac{\partial f}{\partial x} = \tilde{M}$ & $\frac{\partial f}{\partial y} = \tilde{N}$

$\Rightarrow f = \int 2xy^2 + 3x^2\,dx = x^2y^2 + x^3 + g(y)$. $\frac{\partial f}{\partial y} = \tilde{N} \Rightarrow$

$\cancel{2x^2y} + \frac{\partial g}{\partial y} = \cancel{2x^2y} \Rightarrow g = c$. ∴ $f = x^2y^2 + x^3 + c$.

∴ An implicit solution is $x^2y^2 + x^3 = c$.

39. ⓐ Show that a 1-par. family of solutions of the eq'n $(4xy+3x^2)\,dx + (2y+2x^2)\,dy = 0$ is $x^3+2x^2y+y^2=c$.

Implicitly differentiating $x^3+2x^2y+y^2=c$:

$$3x^2 + 4xy + 2x^2\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow (3x^2+4xy)\,dx + (2x^2+2y)\,dy = 0.$$

ⓑ Show that the initial conditions $y(0)=-2$ & $y(1)=1$ determine the same implicit solution.

$y(0)=-2 \Rightarrow c=4$

& $y(1)=1 \Rightarrow c=4$. So, both same implicit solution

$$x^3+2x^2y+y^2=4.$$

ⓒ Find explicit solutions $y_1(x)$ & $y_2(x)$ s.t. $y_1(0)=-2$ & $y_2(1)=1$

Solving $x^3+2x^2y+y^2=4$ for y:

$$y^2+(2x^2)y+(x^3-4)=0 \Rightarrow y = \frac{-2x^2 \pm \sqrt{4x^4-4(x^3-4)}}{2}$$

$$= -x^2 \pm \sqrt{x^4-x^3+4}.$$

Let $y_1(x) = -x^2 - \sqrt{x^4-x^3+4}$.
$y_1(0) = -2$. ✓

Let $y_2(x) = -x^2+\sqrt{x^4-x^3+4}$.
$y_2(1) = -1+2 = 1$. ✓

44. True or False: Every separable 1st-order eq. $\frac{dy}{dx} = g(x)h(y)$ is exact.

Rewrite as: $\underbrace{\frac{1}{h(y)}}_{N} dy \underbrace{- g(x)}_{M} dx = 0.$

$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} \Rightarrow$ exact. $\therefore$ True