2.3: Linear Eqⁿ's:

25, 27, 33, 46, 48

25, 27, 33: Solve the IVP & give the largest interval I over which the solution is defined.

25. $\frac{dy}{dx} = x + 5y$, $y(0) = 3$.

Recall: • A First-order linear DE has the form
$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

• Its standard form is $\boxed{\frac{dy}{dx} + \underbrace{\frac{a_0(x)}{a_1(x)}}_{P(x)} y = \underbrace{\frac{g(x)}{a_1(x)}}_{F(x)}}$.

i.e., $\frac{dy}{dx} + P(x)y = F(x)$.

• To solve, we want to integrate both sides of
$$\frac{d}{dx}\left[e^{\int P(x)\,dx}\,y\right] = e^{\int P(x)\,dx}F(x)$$
& solve for y.

Here, putting our DE in standard form:

$\frac{dy}{dx} + \underbrace{(-5)}_{P(x)}y = \underbrace{x}_{F(x)}$. $\int P(x)\,dx = \int -5\,dx = -5x$.

$u = x \quad v = -\frac{1}{5}e^{-5x}$
$du = dx \quad dv = e^{-5x}$

$e^{-5x}y = \int xe^{-5x}$

$e^{-5x}y = -\frac{1}{5}xe^{-5x} + \frac{1}{5}\int e^{-5x}\,dx$

$e^{-5x}y = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + c$.

$\therefore y = -\frac{1}{5}x - \frac{1}{25} + ce^{5x}$.

$y(0) = 3 \Rightarrow 3 = -\frac{1}{25} + c$
$\Rightarrow c = \frac{76}{25}$.

$\therefore y = -\frac{1}{5}x - \frac{1}{25} + \frac{76}{25}e^{5x}$.

y is defined on $(-\infty, \infty)$.

$y' = -\frac{1}{5} + \frac{76}{5} e^{5x}$ cont. on $(-\infty, \infty)$.

$\therefore I = (-\infty, \infty)$.

27. $xy' + y = e^x$, $y(1) = 2$.

$$\frac{dy}{dx} + \underbrace{\frac{1}{x}}_{P(x)} y = \underbrace{\frac{1}{x} e^x}_{F(x)}$$

$$\int P(x)\,dx = \int \frac{1}{x}\,dx = \ln|x|.$$

$$e^{\ln(x)} y = \int e^{\ln(x)} \left(\frac{1}{x} e^x\right) dx$$

$$xy = \int e^x\,dx$$

$$xy = e^x + C$$

$$y = \frac{1}{x} e^x + \frac{C}{x}.$$

$y(1) = 2$

$\Rightarrow 2 = e^1 + C$

$\Rightarrow C = 2 - e^1.$

$\therefore y = \frac{1}{x} e^x + \frac{2 - e^1}{x}.$

To find the largest interval where the solution is defined, we need the largest I s.t. y is C^1 on I. (i.e. y defined on I & its der. is cont.).

y defined everywhere except at $y = 0$.

$y' = \ln(x) e^x + \frac{1}{x} e^x + (2 - e^1) \ln(x)$ defined & cont. for $x > 0$. $\therefore I = (0, \infty)$.

33. $(x+1)\frac{dy}{dx} + y = \ln x$, $y(1) = 10$.

$$\frac{dy}{dx} + \frac{y}{x+1} = \underbrace{\frac{\ln x}{x+1}}_{F(x)}$$

$$\int P(x)\,dx = \int \frac{1}{x+1}\,dx = \ln(x+1).$$

$$e^{\ln(x+1)} y = \int e^{\ln(x+1)} \frac{\ln x}{x+1}\,dx$$

$$(x+1)y = \int \frac{(x+1)}{(x+1)} \ln(x)\,dx$$

$\frac{d}{dx}(x\ln x - x)$
$= \ln x + 1 - 1$
$= \ln x.$

$$(x+1)y = x\ln x - x + c$$

$$y = \frac{x\ln x - x}{(x+1)} + \frac{c}{x+1}.$$

$y(1) = 10 \Rightarrow$

$$10 = \frac{-1}{2} + \frac{c}{2}$$

$$20 = -1 + c$$

$$c = 21.$$

$$\therefore y = \frac{x\ln x - x}{(x+1)} + \frac{21}{x+1}.$$

Defined for $x > 0$.

not really rec. to compute this if you can just see that der. will have $\ln x$'s & $(x+1)$'s in denominator. *

$$y' = \frac{d}{dx}\, x\ln x\,(x+1)^{-1} + 21(x+1)^{-1} - x(x+1)^{-1}$$

$$= [\ln x + 1](x+1)^{-1} + x\ln x[-(x+1)^{-2}] + \frac{21(-1)}{(x+1)^2} - (x+1)^{-1} + \frac{x}{(x+1)^2}$$

cont. for $x > 0$. $\therefore I = (0, \infty)$.

46. Reread Example 4 + then find the general solution of the DE on the interval $(-3,3)$.

$(x^2-9)\frac{dy}{dx} + xy = 0.$

$$y' + \underbrace{\frac{x}{x^2-9}}_{P(x)} y = \underbrace{0}_{f(x)}$$

$u = x^2-9$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$\int P(x)\,dx = \int \frac{x}{x^2-9}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|x^2-9|.$

on the interval $(-3,3)$, $x^2-9<0 \Rightarrow \frac{1}{2}\ln|x^2-9| = \frac{1}{2}\ln(-x^2+9) = \ln\sqrt{9-x^2}.$

$e^{\ln\sqrt{9-x^2}}\, y = \int e^{\ln\sqrt{9-x^2}}(0)$

$\Rightarrow \sqrt{9-x^2}\, y = 0 + C$

$\Rightarrow y = \frac{C}{\sqrt{9-x^2}}.$

$P(x)$ & $f(x)$ are cont. on $(-3,3)$, so this is the general solution on $(-3,3)$.

48. Reread example 6 & then discuss why it's technically incorrect to say that $y = \begin{cases} 1-e^{-x}, & 0 \le x \le 1 \\ (e-1)e^{-x}, & x > 1 \end{cases}$ is a "solution" of the IVP on the interval $[0,\infty)$.

In order for this to be a "solution", we need y to be C' on $[0,\infty)$.

But notice that y is not diff. at $x=1$:

$$\lim_{x\to 1^-} \frac{y(x)-y(1)}{x-1} = -e^{-1}. \qquad \lim_{x\to 1^+} \frac{y(x)-y(1)}{x-1} = (e-1)(-e^{-x})\Big|_{x=1} = (e-1)(-e^{-1}) = \frac{-1}{e}\cdot\frac{1}{Fe}.$$

$\therefore$ not C' b/c der. not defined at $x=1$. Not same.