Ch. 1: Intro to DE's:

1.1: Def^{in}'s & Terminology:

#1, 3, 5, 7, 9, 15, 25, 26, 33, 35, 43, 46

$\boxed{1\text{-}7}$: State order. Linear?

1. $(1-x)y'' - 4xy' + 5y = \cos x$

Recall: • The order of a DE is the order of the highest derivative in the eq^{in}.

• A DE $F(x, y, y', \ldots, y^{(n)}) = 0$ is linear if F is linear in $y, y', \ldots, y^{(n)}$; i.e.,
$F(x, y, \ldots, y^{(n)}) = a_n(x)y^{(n)} + \cdots + a_0(x)y - g(x) = 0.$

Here it has order 2 & is linear.

3. $t^5 y^{(4)} - t^3 y''' + 6y = 0.$

order 4 & linear.

5. $\dfrac{\partial^2 y}{\partial x^2} = \sqrt{1 + \left(\dfrac{\partial y}{\partial x}\right)^2}.$

Order 2 & nonlinear; b/c of the $\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}$ term.

7. $(\sin\theta)y''' - (\cos\theta)y' = 2.$

order 3 & linear.

9. Is $(y^2-1)\,dx + x\,dy = 0$ linear in y? in x?

To check linear in y, want x to be independent variable (i.e. write in form $\frac{dy}{dx}$): $(y^2-1) + x\frac{dy}{dx} = 0$.
~~Not~~ Not linear in y, b/c y^2 term.

To check in x: $(y^2-1)\frac{dx}{dy} + x = 0$. Linear in x.

15. ⓐ Verify $(y-x)y' = y-x+8$; $y = x+4\sqrt{x+2}$ is an explicit solution to the DE. ⓑ What is the domain of y as a function? ⓒ Domain of y as a solution (give an interval of def'n).

Recall: y is a solution of an n^{th}-order DE if y satisfies both sides of the DE AND $\exists$ an interval I s.t. y is C^n on I (i.e. y is defined on I & possesses at least n derivatives on I that are cont.).

ⓐ $y' = \frac{dy}{dx} = 1 + \frac{4}{2}(x+2)^{-\frac{1}{2}} = 1 + \frac{2}{\sqrt{x+2}}$.

$(y-x)y' = \left[(x+4\sqrt{x+2})-x\right]\left(1+\frac{2}{\sqrt{x+2}}\right)$

$= 4\sqrt{x+2} + 8$. Same ✓

$y-x+8 = (x+4\sqrt{x+2}) - x + 8 = 4\sqrt{x+2} + 8$.

ⓑ $y = x+4\sqrt{x+2}$ is defined so long as $x+2 \geq 0$ $\Leftrightarrow x \geq -2$. $\therefore$ Domain of y as a function is $\{x \in \mathbb{R} \mid x \geq -2\}$ "$= [-2,\infty)$".

ⓒ Our DE is first order, so we need an interval where y is C^1. $y' = 1 + \frac{2}{\sqrt{x+2}}$ is ~~[illegible]~~ defined

For $x \geq -2$ & so long as $x+2 \neq 0 \Leftrightarrow x \neq -2$.
$\therefore$ Defined for $x > -2$. $\therefore$ It's cont. on the interval $(-2, \infty)$. This is the interval of defin.

$\boxed{33-35}$: Use $y = c$ for $-\infty < x < \infty$ is a constant function $\Leftrightarrow y' = 0$ to determine whether the DE possesses constant solutions.

33. $3xy' + 5y = 10$

$3xy' = 10 - 5y \Leftrightarrow y' = \dfrac{10-5y}{3x} = 0 \Leftrightarrow 10 = 5y \Leftrightarrow y = 2.$

i.e. $y = c$ is a constant solution $\Rightarrow 3xy' = 10 - 5y$ & $y' = 0$.
If $y = 2 \Rightarrow y' = 0$ & $3x(0) = 10 - 5(2) \Rightarrow 0 = 0$. ✓
$\therefore$ DE possesses constant solution $y = 2$.

35. $(y-1)y' = 1$

If this DE has a constant solution $\Rightarrow y' = 0 \Rightarrow (y-1)(0) = 1$
$\Rightarrow 0 = 1$. contradiction $\therefore$ no constant solutions.

25. Verify that the piece-wise-defined function
$$y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$
is a solution of the DE $xy' - 2y = 0$ on $(-\infty, \infty)$.

The DE is 1^{st}-order, so must show y satisfies DE & is C^1 on $(-\infty,\infty)$.

$\underline{x<0}$: $\frac{d}{dx}(-x^2) = -2x$. $\quad xy' - 2y = x(-2x) - 2(-x^2) = -2x^2 + 2x^2 = 0.$ ✓

$\underline{x \geq 0}$: $\frac{d}{dx}(x^2) = 2x$. $\quad xy' - 2y = x(2x) - 2(x^2) = 0.$ ✓

Need to check if y' continuous on $(-\infty,\infty)$.

$$y' = \begin{cases} -2x, & x < 0 \\ 2x, & x \geq 0. \end{cases} \qquad \lim_{x \to 0^-} y' = \lim_{x \to 0} -2x = 0.$$

$$\lim_{x \to 0^+} y' = \lim_{x \to 0} 2x = 0.$$

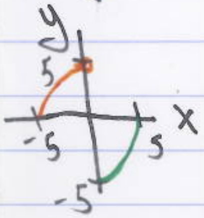
same!

$\therefore$ DE C^1 on $(-\infty,\infty) \Rightarrow y$ is a solution on $(-\infty,\infty)$.

26. In Ex. 5 we saw $y = \sqrt{25-x^2}$ & $y = -\sqrt{25-x^2}$ are solutions of $y' = \frac{-x}{y}$ on $(-5,5)$. Explain why

$$y = \begin{cases} \sqrt{25-x^2} & -5 < x < 0 \\ -\sqrt{25-x^2} & 0 \leq x < 5 \end{cases}$$

not a solution to DE on $(-5,5)$.



$$\cancel{y' = \begin{cases} \frac{1}{2}(25-x^2)^{-\frac{1}{2}} \cdot -2x & -5 < x < 0 \\ -\frac{1}{2}(25-x^2)^{-\frac{1}{2}} \cdot -2x & 0 \leq x < 5 \end{cases}}$$

$$\cancel{= \begin{cases} -x(25-x^2)^{-\frac{1}{2}} & -5 < x < 0 \\ x(25-x^2)^{-\frac{1}{2}} & 0 \leq x < 5. \end{cases}}$$

Not necessary!

We can see y is not cont. @ 0:

$\lim_{x \to 0^-} y = 5.$

$\lim_{x \to 0^+} y = -5.$

not same

Recall:
Differentiable at $x \Rightarrow$ cont. @ x.
$\therefore$ not cont @ $x \Rightarrow$ not diff. @ x.

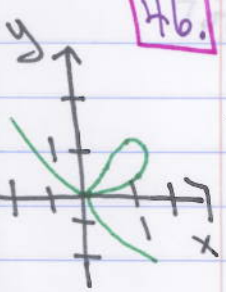
y not cont. @ $x=5 \Rightarrow y$ not diff. @ $x=0$
$\Rightarrow y'$ not defined @ $x=0$.
$\therefore$ y is not a solution on $(-5,5)$, b/c y not C' on this interval.

43. Given that $y=\sin x$ is an explicit solution of the 1st-order DE $y'=\sqrt{1-y^2}$, Find an interval of def'n.

We need to find an interval where y is C', & the DE is satisfied.
$y=\sin x$ defined everywhere, & $y'=\cos x$ cont..

$y'=\sqrt{1-y^2}=\sqrt{1-\sin^2 x}=\sqrt{\cos^2 x}=|\cos x|$

$y=\sin x \Rightarrow y'=\cos x$. So, $\cos x=|\cos x|$ when $\cos x \geq 0$.
A possible interval would be $(-\frac{\pi}{2},\frac{\pi}{2})$.



46. The given graph represents the graph of an implicit solution $G(x,y)=0$ of a DE $y'=F(x,y)$. The relation $G(x,y)=0$ implicitly defines several solutions of the DE. Mark off segments of the corr. to graphs of solutions. Estimate an interval of def'n of each solution ϕ.

We need each ϕ to be a function & differentiable.
function: $\phi(x)=y$... each x maps to one y value.
differentiable: can't have slope of ∞