Math 2C03 - Assignment #5 [Written Part]

[3pts] 1. Evaluate the improper integral $\int_0^\infty t^2 e^{-4t}\cos(2t)\,dt$ using the Laplace transform.

By def'n, $\mathcal{L}\{t^2\cos(2t)\} = \int_0^\infty e^{-st}\, t^2\cos(2t)\,dt = F(s)$.

So, $F(4) = \int_0^\infty e^{-4t}\, t^2\cos(2t)\,dt = \mathcal{L}\{t^2\cos(2t)\}\Big|_{s=4}$

$$= \left[\frac{d^2}{ds^2}(-1)^2\,\mathcal{L}\{\cos(2t)\}\right]\Bigg|_{s=4} = \left[\frac{d^2}{ds^2}\,\frac{s}{s^2+4}\right]\Bigg|_{s=4}$$

$$= \left[\frac{d}{ds}\,\frac{(s^2+4)-2s^2}{(s^2+4)^2}\right]\Bigg|_{s=4} = \frac{(2s-4s)(s^2+4)^2-2(-s^2+4)(s^2+4)(2s)}{(s^2+4)^4}\Bigg|_{s=4}$$

$$= \frac{-2(4)(20)^2-2(-12)(20)(8)}{20^4} = \frac{-8(20)+24(8)}{20^3} = \frac{8(4)}{8000} = \frac{4}{1000}$$

$= 1/250$. $\therefore F(4) = \int_0^\infty t^2 e^{-4t}\cos(2t)\,dt = 1/250$.

[3pts] 2.(a) Solve the system of DE's: $\begin{cases} x' + x + 4y = 10 \\ x - y' - y = 0 \end{cases}$, $x(0)=4$, $y(0)=3$.

$$\begin{cases} [sX(s) - \overbrace{x(0)}^{4}] + X(s) + 4Y(s) = 10/s \\ X(s) - [sY(s) - \underbrace{y(0)}_{3}] - Y(s) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} X(s)[s+1] + 4Y(s) = \frac{10}{s}+4 \\ X(s) + [-s-1]Y(s) = -3 \end{cases} \Rightarrow \begin{bmatrix} s+1 & 4 \\ 1 & -s-1 \end{bmatrix}\begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \frac{10}{s}+4 \\ -3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} -s-1 & -4 \\ -1 & s+1 \end{bmatrix} \frac{1}{-(s+1)^2-4} \begin{bmatrix} \frac{10}{s}+4 \\ -3 \end{bmatrix} = \frac{-1}{(s+1)^2+4}\begin{bmatrix} -10-4s-\frac{10}{s}+8 \\ -\frac{10}{s}-7-3s \end{bmatrix}$$

$$\Rightarrow X(s) = \frac{\frac{10}{s} + 4s + 2}{(s+1)^2+4} = \frac{4s^2 + 10 + 2s}{s[(s+1)^2+4]}$$

$$\& \quad Y(s) = \frac{3s + 7 + \frac{10}{s}}{(s+1)^2+4} = \frac{3s^2 + 7s + 10}{s[(s+1)^2+4]}.$$

$$\frac{4s^2+10+2s}{s[(s+1)^2+4]} = \frac{A_1}{s} + \frac{B_1 s + C_1}{(s+1)^2+4} \Rightarrow 4s^2+10+2s = A_1((s+1)^2+4) + B_1 s^2 + C_1 s$$

$$\Rightarrow 10 = 5A_1 \Rightarrow A_1 = 2 \Rightarrow 4s^2+2s+10 = 2(s^2+2s+5) + B_1 s^2 + C_1 s$$

$$\Rightarrow 4 = 2 + B_1 \ \& \ 2 = 4 + C_1 \ \& \ 10 = 10 \Rightarrow B_1 = 2 \ \& \ C_1 = -2$$

$$\Rightarrow X(s) = \frac{2}{s} + \frac{2s}{(s+1)^2+4} - \frac{2}{(s+1)^2+4} = \frac{2}{s} + \frac{2(s+1)}{(s+1)^2+4} - \frac{2}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}$$

$$\Rightarrow \underline{X(t)} = \underline{2 + 2e^{-t}\cos(2t) - 2e^{-t}\sin(2t)}.$$

$$\frac{3s^2+7s+10}{s[(s+1)^2+4]} = \frac{A_1}{s} + \frac{B_1 s + C_1}{(s+1)^2+4} \Rightarrow 3s^2+7s+10 = A_1[(s+1)^2+4] + B_1 s^2 + C_1 s$$

$$\Rightarrow 10 = 5A_1 \Rightarrow A_1 = 2 \Rightarrow 3s^2 + 7s + 10 = 2s^2 + 4s + 10 + B_1 s^2 + C_1 s$$

$$\Rightarrow 3 = 2 + B_1 \ \& \ 7 = 4 + C_1 \ \& \ 10 = 10 \Rightarrow B_1 = 1 \ \& \ C_1 = 3$$

$$\Rightarrow Y(s) = \frac{2}{s} + \frac{s}{(s+1)^2+4} + \frac{3}{(s+1)^2+4} = \frac{2}{s} + \frac{s+1}{(s+1)^2+4} + \frac{2}{(s+1)^2+4}$$

$$\Rightarrow \underline{y(t)} = \underline{2 + e^{-t}\cos(2t) + e^{-t}\sin(2t)}.$$

$\therefore$ The solution is $\begin{cases} x(t) = 2 + 2e^{-t}\cos(2t) - 2e^{-t}\sin(2t) \\ y(t) = 2 + e^{-t}\cos(2t) + e^{-t}\sin(2t). \end{cases}$

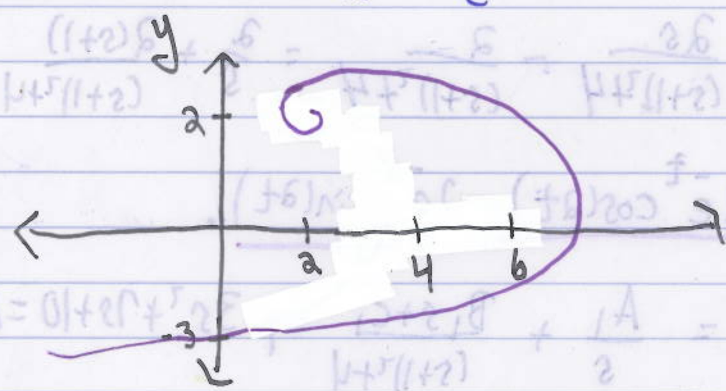
[2pts] $\boxed{b}$ The solution $\begin{cases} x(t) = 2 + 2e^{-t}\cos(2t) - 2e^{-t}\sin(2t) \\ y(t) = 2 + e^{-t}\cos(2t) + e^{-t}\sin(2t) \end{cases}$

parametrizes a curve $(x(t), y(t))$ in $\mathbb{R}^2$.

$\lim\limits_{t\to\infty} (x(t), y(t)) = (2,2)$, since

$$\lim_{t\to\infty} e^{-t}\cos(2t) = \lim_{t\to\infty} \frac{\cos(2t)}{e^{t}} = 0$$

& $\lim\limits_{t\to\infty} e^{-t}\sin(2t) = \lim\limits_{t\to\infty} \frac{\sin(2t)}{e^{t}} = 0.$



Math 2C03 - Assignment #5 [WeBWorK]

1. Find a solution to $\begin{cases} x' = y - x + t \\ y' = y \end{cases}$, $x(0) = 2$, $y(0) = 8$.

$$\begin{cases} \mathcal{L}\{x'\} = \mathcal{L}\{y\} - \mathcal{L}\{x\} + \mathcal{L}\{t\} \\ \mathcal{L}\{y'\} = \mathcal{L}\{y\} \end{cases}$$

$$\Rightarrow \begin{cases} [sX(s) - \overset{2}{x(0)}] = Y(s) - X(s) + \frac{1}{s^2} \\ [sY(s) - \underset{8}{y(0)}] = Y(s) \end{cases}$$

$$\Rightarrow \begin{cases} X(s)[s+1] - Y(s) = \frac{1}{s^2} + 2 \\ [s-1]Y(s) = 8 \end{cases} \Rightarrow \begin{bmatrix} s+1 & -1 \\ 0 & s-1 \end{bmatrix} \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2} + 2 \\ 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} = \frac{1}{(s+1)(s-1)} \begin{bmatrix} s-1 & 1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} \frac{1}{s^2} + 2 \\ 8 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{s^2} + 2(s-1) + 8 \\ 8(s+1) \end{bmatrix} \frac{1}{(s+1)(s-1)}$$

$$\Rightarrow X(s) = \frac{1}{s^2(s+1)} + \frac{2}{s+1} + \frac{8}{(s+1)(s-1)} \quad \& \quad Y(s) = \frac{8}{s-1}$$

$$\Rightarrow x(t) = 2e^{-t} + 8\mathcal{L}^{-1}\left\{\frac{1}{(s+1)(s-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)}\right\} \;\&\; y(t) = 8e^{t}.$$

$$\frac{1}{s^2(s+1)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{A_3}{(s+1)} \Rightarrow 1 = A_1 s(s+1) + A_2(s+1) + A_3 s^2$$

$$\Rightarrow 1 = A_2 \;\&\; 1 = A_3 = 1 \Rightarrow 1 = A_1 s(s+1) + s + 1 + s^2 \Rightarrow 1 = s^2[1 + A_1] + s[1 + A_1] + 1$$

$$\Rightarrow A_1 = -1. \text{ So } \frac{1}{s^2(s+1)} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{(s+1)}.$$

$$\frac{1}{(s+1)(s-1)} = \frac{A_1}{s+1} + \frac{A_2}{s-1} \Rightarrow 1 = A_1(s-1) + A_2(s+1) \Rightarrow 1 = 2A_2 \Rightarrow A_2 = \tfrac{1}{2}$$
$$\& \Rightarrow 1 = -2A_1 \Rightarrow A_1 = -\tfrac{1}{2}.$$

$$\therefore x(t) = 2e^{-t} + 8\mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\Rightarrow x(t) = 2e^{-t} - 4e^{-t} + 4e^{t} - 1 + t + e^{-t}$$
$$= -e^{-t} + 4e^{t} + t - 1$$

$\&$ $y(t) = 8e^{t}$.

2. Find the solution to $\begin{cases} x' = y \\ y' = 18x + 3y \end{cases}$, $x(0) = 0$, $y(0) = 2$.

$$\begin{cases} sX(s) - \overbrace{x(0)}^{0} = Y(s) \\ sY(s) - \underbrace{y(0)}_{2} = 18X(s) + 3Y(s) \end{cases}$$

$$\Rightarrow \begin{cases} X(s) = \frac{1}{s} Y(s) \\ Y(s)[s-3] - 18X(s) = 2 \end{cases} \Rightarrow Y(s)\left[s - 3 - \frac{18}{s}\right] = 2$$

$$\Rightarrow Y(s) = \frac{2}{\frac{s^2 - 3s - 18}{s}} = \frac{2s}{(s-6)(s+3)}$$

$$\Rightarrow X(s) = \frac{2}{(s-6)(s+3)}.$$

$$\frac{1}{(s-6)(s+3)} = \frac{A_1}{s-6} + \frac{A_2}{s+3} \Rightarrow 1 = A_1(s+3) + A_2(s-6) \Rightarrow 1 = -9A_2$$
$$\Rightarrow A_2 = -\frac{1}{9} \ \& \ 1 = 9A_1 \Rightarrow A_1 = \frac{1}{9}.$$

So, $\frac{2}{(s-6)(s+3)} = \frac{2}{9(s-6)} - \frac{2}{9(s+3)}$.

$$\frac{2s}{(s-6)(s+3)} = \frac{A_1}{s-6} + \frac{A_2}{s+3} \Rightarrow 2s = A_1(s+3) + A_2(s-6) \Rightarrow 12 = 9A_1 \Rightarrow A_1 = \frac{4}{3}$$
$$\& \Rightarrow -6 = -9A_2 \Rightarrow A_2 = \frac{2}{3}.$$

$$\therefore x(t) = \frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\} - \frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = \frac{2}{9}e^{6t} - \frac{2}{9}e^{-3t}$$

$$y(t) = \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = \frac{4}{3}e^{6t} + \frac{2}{3}e^{-3t}.$$

3. let $f(t) = e^{t}$ & $g(t) = e^{-2t}$, defined on $0 \le t < \infty$. Compute $f*g$ in 2 different ways:

(a) $(f*g)(t) = \int_0^t f(w)\,g(t-w)\,dw = \int_0^t e^{w} e^{-2(t-w)}\,dw$

$= \int_0^t e^{-2t+3w}\,dw = \int_0^t e^{-2t} e^{3w}\,dw = \frac{1}{3} e^{-2t} e^{3w} \Big|_0^t = \frac{1}{3} e^{t} - \frac{1}{3} e^{-2t}.$

(b) $(f*g)(t) = \mathcal{L}^{-1}\{F(s)\,G(s)\} = \mathcal{L}^{-1}\{\mathcal{L}\{e^t\}\,\mathcal{L}\{e^{-2t}\}\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s-1}\,\frac{1}{s+2}\right\} = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \frac{1}{3}e^{t} - \frac{1}{3}e^{-2t}.$

$\frac{1}{(s-1)(s+2)} = \frac{A_1}{s-1} + \frac{A_2}{s+2} \Rightarrow 1 = (s+2)A_1 + (s-1)A_2 \Rightarrow A_1 = \frac{1}{3}$ & $A_2 = -\frac{1}{3}.$

4. Consider the integral eq'n $\int_0^t \sin(6(t-w))\,y(w)\,dw = 7t^2$, defined for $t \ge 0$.

(a) Use convolution & Laplace Transforms to find $Y(s)$:

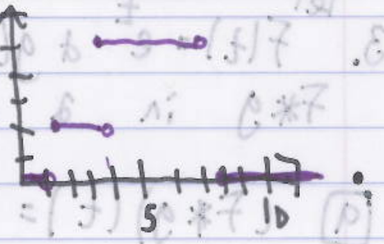
$f(t) = y(t),\ g(t) = \sin(6t) \Rightarrow f*g = \int_0^t y(w)\sin(6(t-w))\,dw.$

So, $\mathcal{L}\{f*g\} = \mathcal{L}\{7t^2\} \Rightarrow \mathcal{L}\{y(t)\}\,\mathcal{L}\{\sin(6t)\} = \frac{7\cdot 2}{s^3}$

$\Rightarrow Y(s)\cdot\frac{6}{s^2+36} = \frac{14}{s^3} \Rightarrow Y(s) = \frac{7[s^2+36]}{3\ s^3} = \frac{7}{3}\frac{1}{s} + \frac{84}{s^3}$

$\Rightarrow y(t) = \frac{7}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 42\,\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{7}{3} + 42t^2.$

5. The graph of $F(t)$ is given by



(a) Represent $F(t)$ using a combination of Heaviside functions $h(t-a)$.

$F(t) = 2h(t-1) + 3h(t-4) - 5h(t-8)$.

(b) Find the Laplace transform $F(s) = \mathcal{L}\{F(t)\}$ for $s \neq 0$.

$\mathcal{L}\{F(t)\} = 2\mathcal{L}\{h(t-1)\} + 3\mathcal{L}\{h(t-4)\} - 5\mathcal{L}\{h(t-8)\}$

$= \dfrac{2e^{-s} + 3e^{-4s} - 5e^{-8s}}{s}$.

6. Find the radius of convergence of the power series.

$\sum_{n=1}^{\infty} \dfrac{(-1)^n (x+12)^n}{9^n n^7}$.

Ratio Test: $\lim_{n\to\infty} \left|\dfrac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \left|\dfrac{\frac{(-1)^{n+1}}{9^{n+1}(n+1)^7}}{\frac{(-1)^n}{9^n n^7}}\right|$

$= \lim_{n\to\infty} \left|\dfrac{9^n n^7}{9^{n+1}(n+1)^7}\right| = \lim_{n\to\infty} \left|\dfrac{n^7}{9(n+1)^7}\right| = \dfrac{1}{9} \Rightarrow R = \dfrac{1}{1/9} = 9.$

7. Use the Laplace Transform to solve the IVP
$y'' + 10y' + 74y = \delta(t-8)$, $y(0) = 0$, $y'(0) = 0$.

$s^2 Y(s) - s\overbrace{y(0)}^{0} - \overbrace{y'(0)}^{0} + 10[sY(s) - \overbrace{y(0)}^{0}] + 74Y(s) = e^{-8s}$

$\Rightarrow Y(s)[s^2 + 10s + 74] = e^{-8s} \Rightarrow Y(s) = \dfrac{e^{-8s}}{(s+5)^2 + 49}$

$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-8s}}{(s+5)^2+49}\right\} = \mathcal{L}^{-1}\left\{e^{-8s}\mathcal{L}\left\{\frac{e^{-5t}\sin(7t)}{7}\right\}\right\}$

$\mathcal{L}^{-1}\left\{\frac{1}{(s+5)^2+49}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+7^2}\Big|_{s\to s+5}\right\} = \frac{e^{-5t}}{7}\sin(7t)$

$= \frac{1}{7}e^{-5(t-8)}\sin(7(t-8))\,u(t-8).$

8. Consider the DE $2xy'' + (1+x)y' - 2y = 0$. Note that $x_0 = 0$ is a regular singular point. If you were to look for a series solution about $x_0 = 0$ of the form $\sum_{n=0}^{\infty} a_n x^{n+r}$, then the indicial eq'n is given by:

$2xy'' + (1+x)y' - 2y = 0$

$y'' + \underbrace{\frac{1+x}{2x}}_{P(x)} y' - \underbrace{\frac{1}{x}}_{Q(x)} y = 0$

$p(x) = xP(x) = \frac{1+x}{2} = \underbrace{\frac{1}{2}}_{a_0} + \frac{1}{2}x + 0x^2 + 0x^3 + \dots$

$q(x) = x^2 Q(x) = -x = \underbrace{0}_{b_0} - x + 0x^2 + 0x^3 + \dots$

Indicial eq'n given by $r(r-1) + a_0 r + b_0 = 0$

$\Leftrightarrow r(r-1) + \frac{1}{2}r = 0$

$\Leftrightarrow r^2 - r + \frac{1}{2}r = 0$

$\Leftrightarrow r^2 - \frac{1}{2}r = 0$

$\Leftrightarrow r(r - \frac{1}{2}) = 0 \Leftrightarrow r = 0$ or $r = \frac{1}{2}$.

The difference of roots is not an integer, & so one can obtain a linearly independent of the given form for each root in the indicial eq'n. The recurrence formula for the coefficients of the solution w/ the larger root is given by:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}.$$

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}.$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}.$$

$$2xy'' + (1+x)y' - 2y = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n(n+r) x^{n+r-1} + \sum_{n=0}^{\infty} a_n(n+r) x^{n+r}$$

$$-2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$= x^r \Big[\underbrace{\sum_{n=0}^{\infty} 2a_n(n+r)(n+r-1) x^{n-1} + \sum_{n=0}^{\infty} a_n(n+r) x^{n-1}}_{K=n-1} + \underbrace{\sum_{n=0}^{\infty} a_n (n+r) x^{n}}_{K=n}$$

$$\underbrace{-2\sum_{n=0}^{\infty} a_n x^n}_{K=n} \Big] = 0$$

$$= x^r \Big[\sum_{K=-1}^{\infty} \big(2a_{K+1}(K+1+r)(K+r) + a_{K+1}(K+1+r)\big) x^K + \sum_{K=0}^{\infty} \big(a_K(K+r) - 2a_K\big) x^K \Big]$$

$$= x^r \Big[2a_0 r(r-1) + a_0 r + \sum_{K=0}^{\infty} \big(2a_{K+1}(K+1+r)(K+r) + a_{K+1}(K+1+r) + a_K(K+r) - 2a_K\big) x^K \Big]$$

$$\Rightarrow 2r^2 - r = 0 \ \ \& \ \ [2(K+1+r)(K+r) + (K+1+r)]a_{K+1} + (K+r-2)a_K = 0$$

Let $n = K+1$ $\Rightarrow a_n = \dfrac{-(n+r-3)}{2(n+r)(n-1+r) + (n+r)} a_{n-1}.$

$r = \frac{1}{2}$: $a_n = \dfrac{-n + \frac{5}{2}}{(2n+1)(n-\frac{1}{2}) + (n+\frac{1}{2})} a_{n-1}.$

Taking $a_0 = 1$, the first 3 terms of a particular solution is: $a_0 = 1 \Rightarrow a_1 = \dfrac{-1+5/2}{3(\frac{1}{2})+\frac{3}{2}}\underbrace{a_0}_{1} = \dfrac{3/2}{6/2} = \dfrac{3}{6} = \dfrac{1}{2}.$

$a_2 = \dfrac{-2+5/2}{5(3/2)+5/2}\, a_1 = \dfrac{\frac{1}{2}}{10}\cdot\dfrac{1}{2} = \dfrac{1}{40}.$

$y = \sum_{n=0}^{\infty} a_n x^{n+\frac{1}{2}} = x^{\frac{1}{2}} + \frac{1}{2}x^{\frac{3}{2}} + \frac{1}{40}x^{\frac{5}{2}} + \cdots.$

9. Consider the DE $(1+2x^2)y'' - 5xy' + 3y = 0.$
If you were to look for a power series solution about $x_0 = 0$, i.e., of the form $\sum_{n=0}^{\infty} a_n x^n$, then

$a_2 = \dfrac{-3/2}{} a_0$

$a_3 = \dfrac{1/3}{} a_1.$

$y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}.$

$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}.$

$0 = (1+2x^2)y'' - 5xy' + 3y = \underbrace{\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}}_{K=n-2} + \underbrace{2\sum_{n=2} a_n n(n-1) x^n}_{K=n}$

$\underbrace{-5\sum_{n=1}^{\infty} a_n n x^n}_{K=n} + \underbrace{3\sum_{n=0}^{\infty} a_n x^n}_{K=n}$

$= \sum_{K=0}^{\infty} a_{K+2}(K+2)(K+1)x^K + \sum_{K=2}^{\infty} 2a_K K(K-1)x^K - 5\sum_{K=1}^{\infty} a_K K x^K + 3\sum_{K=0} a_K x^K$

$= \underline{2a_2} + 6a_3 x - 5a_1 x + \underline{3a_0} + 3a_1 x + \sum_{K=2}^{\infty}\left[a_{K+2}(K+2)(K+1) + 2a_K K(K-1) - 5a_K K + 3a_K\right] x^K$

$\Rightarrow 3a_0 + 2a_2 = 0$ & $-2a_1 + 6a_3 = 0$

& $(K+2)(K+1)a_{K+2} = -\left[2K(K-1) - 5K + 3\right] a_K$

$\Rightarrow a_{K+2} = \dfrac{-2K(K-1)+5K-3}{(K+2)(K+1)}$ & $a_2 = -\frac{3}{2}a_0$ & $a_3 = \frac{1}{3}a_1.$

The recurrence formula for the coefficients would be given by:

$$a_{n+2} = \frac{-2n(n-1) + 5n - 3}{(n+2)(n+1)} a_n, \ n \geq 2,$$

and the solution of the IVP with $y(0) = 0$, $y'(0) = 2$ would be given by:

$y(0) = 0 \Rightarrow \sum_{n=0}^{\infty} a_n x^n = 0 \Rightarrow a_0 = 0 \Rightarrow$ all even terms are zero.

$y'(0) = 2 \Rightarrow \sum_{n=1}^{\infty} a_n n x^{n-1} = 2 \Rightarrow a_1 = 2 \Rightarrow a_3 = \frac{2}{3}$

$$\Rightarrow a_5 = \frac{-2(3)(2) + 5(3) - 3}{5(3)} = \frac{-12+15-3}{15} = 0$$

$$\Rightarrow a_7 = a_9 = \cdots = 0.$$

$\therefore y(x) = 2x + \frac{2}{3}x^3.$

The first 4 non-zero terms of the solution of the IVP with $y(0) = 5$, $y'(0) = 0$ would be given by:

$y(0) = 5 \Rightarrow \sum_{n=0}^{\infty} a_n x^n = 5 \Rightarrow a_0 = 5.$

$y'(0) = 0 \Rightarrow \sum_{n=1}^{\infty} a_n n x^{n-1} = 0 \Rightarrow a_1 = 0 \Rightarrow$ all odd terms are zero.

$a_2 = \frac{-3}{2} a_0 = \frac{-15}{2}$. $a_4 = \frac{-4(1) + 10 - 3}{4(3)} a_2 = \frac{3}{12} \cdot \left(\frac{-15}{2}\right) = \frac{1}{4} \cdot \frac{-15}{2}$
$= \frac{-15}{8}.$

$$a_6 = -\frac{8(3) + 20 - 3}{6(5)} \left(-\frac{15}{8}\right) = \frac{-7}{30} \cdot \frac{-15}{8} = \frac{-7}{8} \cdot \frac{-1}{2} = \frac{7}{16}.$$

$\therefore y(x) = 5 - \frac{15}{2}x^2 - \frac{15}{8}x^4 + \frac{7}{16}x^6 + \cdots .$

10. Consider the DE $(x^2-9)^2 y''(x) + (x+3)(x-7)y'(x) - (x+7)^2 y(x) = 0$.

List the regular singular points: $\underline{\quad -3 \quad}$,
& the irregular singular points: $\underline{\qquad\qquad}$.

$$y'' + \frac{(x+3)(x-7)}{(x+3)^2(x-3)^2} y' - \frac{(x+7)^2}{(x+3)^2(x-3)^2} y = 0$$

$$\Rightarrow y'' + \underbrace{\frac{x-7}{(x+3)(x-3)^2}}_{P(x)} y' - \underbrace{\frac{(x+7)^2}{(x+3)^2(x-3)^2}}_{Q(x)} y = 0.$$

$x=-3$ & $x=3$ are singular points.

$x+3$ appears to 1^{st} power in $P(x)$ & 2^{nd} power in $Q(x)$
$\Rightarrow x=-3$ regular singular point.

$x-3$ appears to 2^{nd} power in $P(x) \Rightarrow x=3$ irregular singular point.