# Math 2C03 - Assignment #4 [WeBWork]

1. (a) Find the general solution to $y'' + 2y' = 0$.

$m^2 + 2m = 0$

$m(m+2) = 0$

$m = 0,\ m = -2$

$y = A + Be^{-2x}$.

(b) Find the particular solution that satisfies $y(0) = 1$ & $y'(0) = 1$.

$y(0) = 1 \Rightarrow 1 = A + B$

$y'(0) = 1 \Rightarrow 1 = -2Be^{-2(0)} = -2B \Rightarrow B = -\frac{1}{2} \Rightarrow A = \frac{3}{2}$.

$\therefore\ y = \frac{3}{2} - \frac{1}{2}e^{-2x}$.

2. Find the general solution to $y^{(4)} - 8y''' + 12y'' = 0$.

$m^4 - 8m^3 + 12m^2 = 0$

$m^2[m^2 - 8m + 12] = 0$

$m^2(m-2)(m-6) = 0$

$\underbrace{m = 0}_{\text{double}},\ m = 2,\ m = 6$.

$y = A + Bx + Ce^{2x} + De^{6x}$.

3. Suppose the auxiliary eq'n for a linear homog. DE with constant coefficients is $(r-4)^2(r-7)^2 = 0$.

(a) Find such a DE, assuming it's homog. & has constant coef..

$(r-4)^2(r-7)^2 = (r^2 - 8r + 16)(r^2 - 14r + 49)$

$= r^4 - 14r^3 + 49r^2 - 8r^3 + 112r^2 - 392r + 16r^2 - 224r + 784$

$= r^4 - 22r^3 + 177r^2 + 784 - 616r$.

$\therefore\ y'''' - 22y''' + 177y'' - 616y' + 784y = 0$.

49
16
294
490
784

49
8
392

14
8
112

16
14
64
160
224

(b) Find the general solution to this DE.

$r=4$, $r=7$ $\quad y = Ae^{4x} + Bxe^{4x} + Ce^{7x} + Dxe^{7x}$.

$\underbrace{r=4}_{\text{double}}, \underbrace{r=7}_{\text{double}}$

4. For a specific set of fixed constants $a_0, \ldots, a_n$, define $L[y] = a_n y^{(n)}(x) + \cdots + a_0 y(x)$. Consider the nth-order linear DE $L[y] = 20\sin(6x) + 13e^{5x}$. (*)
If it is known that $L[y_1(x)] = 8e^{5x}$ when $y_1(x) = 10xe^{5x}$,
$L[y_2(x)] = 11\sin(6x)$, when $y_2(x) = 11\cos(6x)$,
& $L[y_3(x)] = 10\cos(6x)$, when $y_3(x) = 8\sin(6x)$,
then a solution to (*) is given by:

$L[\frac{13}{8} y_1(x) + \frac{20}{11} y_2(x)] = \frac{13}{8} L[y_1(x)] + \frac{20}{11} L[y_2(x)]$

$= 13e^{5x} + 20\sin(6x)$.

$\therefore\ y = \frac{13}{8}(10xe^{5x}) + \frac{20}{11}(11\cos(6x)) = \frac{65}{4}xe^{5x} + 20\cos(6x)$.

5. Find a particular solution to the 2nd-order linear nonhomog. DE $x^2y'' - 2xy' + 2y = 2\ln(x)$, $x>0$.

Cauchy-Euler: $m(m-1) - 2m + 2 = 0$
$m^2 - 3m + 2 = 0$
$(m-2)(m-1) = 0$
$\Rightarrow m=2,\ m=1$.

$y_c = c_1 \underbrace{x}_{y_1} + c_2 \underbrace{x^2}_{y_2}$.

Variation of Parameters: $y'' - 2x^{-1}y' + 2x^{-2}y = \underbrace{2\ln(x)x^{-2}}_{F(x)}$.

$W = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$.

$$W_1 = \begin{vmatrix} 0 & x^2 \\ 2x^{-2}\ln x & 2x \end{vmatrix} = -2\ln x.$$

$u = \ln x$
$du = \frac{1}{x}dx$
$v = -x^{-1}$
$dv = x^{-2}$

$$u_1 = \int -2x^{-2}\ln x\,dx = -2\left[-\frac{\ln x}{x} + \int x^{-2}\,dx\right]$$

$$= \frac{2\ln x}{x} + 2x^{-1}.$$

$$W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^{-2}\ln x \end{vmatrix} = 2x^{-1}\ln x.$$

$u = \ln x$
$du = \frac{1}{x}dx$
$v = -\frac{1}{2}x^{-2}$
$dv = x^{-3}$

$$u_2 = \int 2x^{-3}\ln x\,dx = 2\left[-\tfrac{1}{2}x^{-2}\ln x + \tfrac{1}{2}\int x^{-3}dx\right]$$

$$= -x^{-2}\ln x - \tfrac{1}{2}x^{-2}.$$

$$y_p = u_1y_1 + u_2y_2 = 2\ln x + 2 \quad \ln x - \tfrac{1}{2}$$
$$= \ln x + 3/2.$$

6. Find a particular solution of the $2^{nd}$-order linear nonhomog. DE $x^2y'' - 2xy' + 2y = 3x^2$, $x > 0$.

From #5, $y_c = c_1x + c_2x^2$.

<u>Variation of Parameters</u>: $y'' - 2x^{-1}y' + 2x^{-2}y = \underbrace{3}_{f(x)}.$

$W = x^2$. $W_1 = \begin{vmatrix} 0 & x^2 \\ 3 & 2x \end{vmatrix} = -3x^2$. $W_2 = \begin{vmatrix} x & 0 \\ 1 & 3 \end{vmatrix} = 3x$.

$u_1 = \int -3\,dx = -3x$. $u_2 = \int 3x^{-1}\,dx = 3\ln x$.

$y_p = -3x^2 + 3x^2\ln x.$

7. Using the method of undetermined coefficients, find a particular solution to $y'' + 7y' + 10y = 3te^{5t}$.

$m^2 + 7m + 10 = 0$ $\quad y_c = c_1 e^{-5t} + c_2 e^{-2t}$.

$(m+5)(m+2) = 0$

$m = -5, \ m = -2$.

$(D-5)^2 (3te^{5t}) = 0$.

$(D-5)^2 (D^2 + 7D + 10)y = 0$

$(m-5)^2 (m+5)(m+2)$

$\underbrace{m = 5}_{\text{double}}, \ m = -5, \ m = -2$

$\therefore y_p = c_1 e^{5t} + c_2 t e^{5t}$

$y_p' = 5c_1 e^{5t} + 5c_2 t e^{5t} + c_2 e^{5t}$

$= (5c_1 + c_2) e^{5t} + 5c_2 t e^{5t}$.

$y_p'' = (25c_1 + 5c_2) e^{5t} + 25c_2 t e^{5t} + 5c_2 e^{5t}$

$= (25c_1 + 10c_2) e^{5t} + 25c_2 t e^{5t}$.

$(25c_1 + 10c_2) e^{5t} + 25c_2 t e^{5t} + (35c_1 + 7c_2) e^{5t} + 35c_2 t e^{5t}$

$+ 10c_1 e^{5t} + 10c_2 t e^{5t} = 3te^{5t}$

$\Rightarrow 70c_1 + 17c_2 = 0 \ \& \ 70c_2 = 3 \Rightarrow c_2 = \frac{3}{70}$

$\begin{array}{r} 17 \\ \times 3 \\ \hline 51 \end{array} \quad \begin{array}{r} 70 \\ \times 70 \\ \hline 4900 \end{array}$

$\Rightarrow c_1 = -\frac{17}{70} c_2 = \frac{-17}{70} \cdot \frac{3}{70} = \frac{-51}{4900}$.

$\therefore y_p = \frac{-51}{4900} e^{5t} + \frac{3}{70} t e^{5t}$.

8. (a) Set up an integral for finding the Laplace transform of $f(t) = t - 10$.

$\int_0^\infty e^{-st} (t - 10)\, dt$.

$u = t$

$du = dt$

$v = -\frac{e^{-st}}{s}$

$dv = e^{-st}$

(b) Find the Laplace transform of $f(t)$.

$\int_0^\infty t e^{-st} dt - 10 \int_0^\infty e^{-st} dt = -\frac{te^{-st}}{s} \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} dt - 10 \int_0^\infty e^{-st} dt$

$$= 0 - 0 + \left(\tfrac{1}{s} - 10\right)(-1)\frac{e^{-st}}{s}\Big|_0^\infty \overset{\text{if } s>0}{=} \left(\frac{1}{s^2} - \frac{10}{s}\right)[0+1]$$

$$\lim_{t\to\infty} -\frac{1}{s}\frac{t}{e^{st}} \overset{\text{L'Hopital's Rule}}{=} -\frac{1}{s}\lim_{t\to\infty}\frac{1}{se^{st}} = 0 \text{ if } s>0.$$

$$= \frac{1}{s^2} - \frac{10}{s}.$$

ⓒ What is the domain of $F(s)$?

$F(s) = \mathcal{L}\{f(t)\} = \frac{1}{s^2} - \frac{10}{s}$ exists if $s > 0$.

9. Consider $f(t) = 3e^{t/3}\sin t$.

ⓐ The function $f(t)$ is: A. cont. on $0 \le t < \infty$.

ⓑ Is $f(t)$ of exponential order $\alpha$ as $t \to \infty$ for some $\alpha$?

Need to check if $\lim_{t\to\infty} \frac{|f(t)|}{e^{\alpha t}} = L$, where $L \ge 0$ is finite.

Let $\alpha = \frac{1}{3}$. Then $\lim_{t\to\infty} \frac{|3e^{\frac{1}{3}t}\sin t|}{e^{\frac{1}{3}t}} = 3\lim_{t\to\infty}\frac{|\sin t|}{e^{2t}} = 0.$

$|\sin t| \le 1$

∴ Yes, of exponential order.

ⓒ Does the Laplace transform of $f(t)$ exist for $s > \alpha$ for some $\alpha > 0$?

Yes, by Theorem 7.1.2, since $f$ is cont. on $0 \le t < \infty$ ⇒ $f$ piecewise cont. on $[0, \infty)$ & $f$ is of exponential order 2, so $\mathcal{L}\{f(t)\}$ exists for $s > 2$.

could have chosen another $\alpha$

10. Use the Laplace transform to solve the following IVP: $y'' - 4y' + 20y = 0$, $y(0) = 0$, $y'(0) = 4$.

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 20\mathcal{L}\{y\} = 0$$

$$\Rightarrow [s^2 Y(s) - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{4}] - 4[sY(s) - \underbrace{y(0)}_{0}] + 20Y(s) = 0$$

$$\Rightarrow Y(s)[s^2 + 20 - 4s] - 4 = 0$$

$$\Rightarrow Y(s)[s^2 - 4s + 20] = 4$$

$$\Rightarrow Y(s) = \frac{4}{s^2 - 4s + 20} = \frac{4}{(s-2)^2 + 16}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+16}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\Big|_{s \to s-2}\right\}$$

1st translation Property

$$\Rightarrow y(t) = e^{2t}\sin(4t).$$

11. Find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$ of the function $F(s) = \frac{4}{s^2+25} - \frac{4s}{s^2+49}$.

$$\mathcal{L}^{-1}\left\{\frac{4}{s^2+25} - \frac{4s}{s^2+49}\right\} = \frac{4}{5}\mathcal{L}^{-1}\left\{\frac{5}{s^2+25}\right\} - 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+49}\right\}$$

$$= \frac{4}{5}\sin(5t) - 4\cos(7t).$$

12. Find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$ of the function $F(s) = \frac{4s-15}{s^2-6s+10}$.

$$\mathcal{L}^{-1}\left\{\frac{4s-15}{s^2-6s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{4(s-3)-3}{(s-3)^2+1}\right\} \underset{\text{1st-translation Theorem}}{=} \mathcal{L}^{-1}\left\{\frac{4s-3}{s^2+1}\Big|_{s\to s-3}\right\}$$

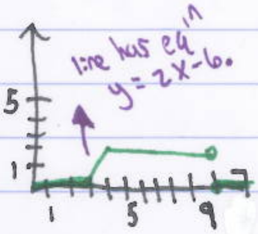
$$= 4\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\Big|_{s\to s-3}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\Big|_{s\to s-3}\right\}$$

$$= 4e^{3t}\cos t - 3e^{3t}\sin t.$$

13. Find the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$ of the function $F(s) = \frac{3e^{-7s}}{s^2+9}$.

$$\mathcal{L}^{-1}\left\{e^{-7s}\frac{3}{s^2+9}\right\} \underset{\text{2nd-Translation Theorem}}{=} \sin(3(t-7))\,h(t-7).$$

14. The graph of $f(t)$ is given below:




(a) Represent $f(t)$ using a combination of Heaviside step functions $h(t-a)$.

$$f(t) = [h(t-3)-h(t-4)](2t-6) + 2[h(t-4)-h(t-9)].$$

(b) Find the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ for $s \neq 0$.

$$\mathcal{L}\{f(t)\} = 2\mathcal{L}\{t\,h(t-3)\} - 6\mathcal{L}\{h(t-3)\} - 2\mathcal{L}\{t(h-4)\} + 6\mathcal{L}\{h(t-4)\} + 2\mathcal{L}\{h(t-4)\} - 2\mathcal{L}\{h(t-9)\}$$

using $\mathcal{L}\{h(t-a)\} = \frac{e^{-as}}{s}$ & $\mathcal{L}\{g(t)h(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$.

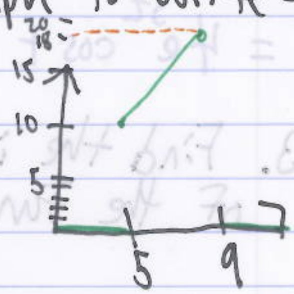
$$= 2e^{-3s}\mathcal{L}\{t+3\} - 6\frac{e^{-3s}}{s} - 2e^{-4s}\mathcal{L}\{t+4\} + 8\frac{e^{-4s}}{s} - 2\frac{e^{-9s}}{s}$$

$$= \frac{2e^{-3s}}{s^2} + \cancel{\frac{6e^{-3s}}{s}} - \cancel{\frac{6e^{-3s}}{s}} - \frac{2e^{-4s}}{s^2} - \cancel{\frac{8e^{-4s}}{s}} + \cancel{\frac{8e^{-4s}}{s}} - \frac{2e^{-9s}}{s}$$

$$= \frac{2e^{-3s}}{s^2} - \frac{2e^{-4s}}{s^2} - \frac{2e^{-9s}}{s}.$$

15. (a) Graph the function $F(t) = 2t\,(h(t-5) - h(t-9))$ for $0 \le t < \infty$. Use your graph to write this piecewise cont. function as:

$$F(t) = \begin{cases} \underline{0} & \text{if } 0 \le t < 5, \\ \underline{2t} & \text{if } 5 \le t < 9, \\ \underline{0} & \text{if } 9 \le t < \infty. \end{cases}$$




(b) Evaluate $F(7)$.

$F(7) = 14.$