Math 2C03 - Assignment #3 [Written part]

[4pts] 1. Solutions of autonomous DE's behave asymptotically (approach a value arbitrarily close) & have the translation property (if $y(x)$ is a solution of an autonomous DE $y' = f(y)$, then $y_1(x) = y(x-k)$ is also a solution $\forall k \in \mathbb{R}$). We briefly discussed why both of these things were true in class, but didn't formally write down the details.
Please give a detailed explanation why both of these facts are true. In your argument, be sure to include why graphs of nonconstant solutions can't cross the graph of a constant solution, can't oscillate or have rel. max./min., & why they must always be increasing or decreasing. (Recall: given $y' = f(y)$, we assume f & f' are cont. on some interval I).

Translation property: We can see this by looking at the direction field of $y' = f(y)$. The slope at any point (x_0, y_0) does not depend on x. Therefore, in the direction field, slopes along a horizontal line are all the same, i.e. $\frac{dy}{dx}(x_0, y_0) = f(y_0) = \frac{dy}{dx}(x_0 - k, y_0)$ for any $k \in \mathbb{R}$. Therefore, given a solution curve $y(x)$, we can translate to the right & left by k, & we'll still have a solution.

More concretely: Suppose $y(x)$ is a solution of $y' = f(y)$. i.e. $\frac{d}{dx} y(x) = f(y(x))$.
Then $\frac{d}{dx} y(x-k) = y'(x-k) = f(y(x-k)) \Rightarrow y(x-k)$ is a solution too. ($f(y)$ not depend on x).

Behave asymptotically:

Non-constant Solutions $\tilde{y}$ won't cross the graph of constant solutions $y = c$ for the following reason: $y' = F(y)$ & F & F' are cont. on $I \Rightarrow \exists$ an interval I_0 around every point in I s.t. a solution exists & is unique on I_0 (Theorem 1.2.1). Suppose $\tilde{y}$ intersects $y = c$ at a point (x_0, c). Then the IVP with $y(x_0) = c$ would have 2 distinct solutions, which is a contradiction. $\therefore$ y & $\tilde{y}$ can't intersect.

Non-constant solutions $\tilde{y}$ must be increasing or decreasing for the following reason: we know $\tilde{y}' = F(\tilde{y})$ is continuous. Therefore the solution is smooth & if it changes from increasing to decreasing (or vice-versa) there must exist a point where $\tilde{y}' = 0$, i.e. $\tilde{y}$ must intersect a constant solution $y = c$. But by the previous paragraph we know this can't happen.

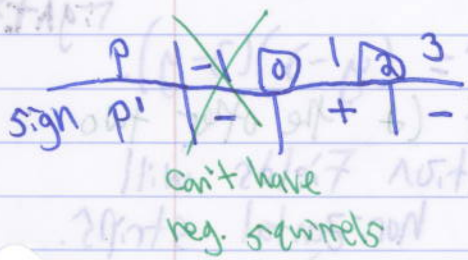
Similarly, if the graph of $\tilde{y}$ oscillated or had a rel. max/min. then it must have a horizontal tangent line at some point, which is contradiction.

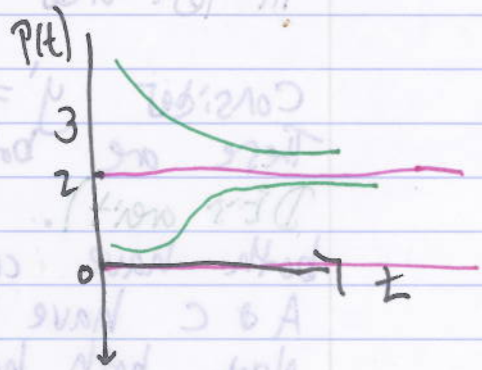
Therefore, for each non-constant solution curve $\tilde{y}$, if it has a constant $\overset{y=c}{\wedge}$ solution above / below it, it must approach c asymptotically.

[3pts] 2. Suppose that the population p (in thousands) of squirrels in Hamilton can be modelled by the DE $\frac{dp}{dt} = p(2-p)$.

(a) If the initial population of squirrels is 3000, what can you say about the long-term behaviour of the squirrel population?

$p(2-p) = 0 \iff p = 0$ or $p = 2$.





Squirrel population will tend to 2000 in the long-term.

(b) Can a population of 1000 ever decline to 500? Explain.

No, b/c solution curves strictly increasing b/w $0 < p < 2$, so if we begin w/ 1000 it'll never decrease to 500.

(c) Can a population of 1000 ever increase to 3000? Explain.

No, b/c a solution curve starting at $p = 1$ can't cross the constant solution $p = 2$, so it will tend to 2000, but can never reach 3000.

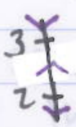
[3pts] 3. Consider the 1st-order DE's $y' = (2-y)(3-y)$, $y' = (y-2)(3-y)$, $y' = (2-x)(3+x)$, $y' = (2-y)(3+x)$.

Assign the direction fields below to the appropriate DE, & write a short paragraph to justify your choices.

* I'm not going to draw the 3 direction fields here, so let me notate them as $\boxed{A}$, $\boxed{B}$, $\boxed{C}$ in the order they appear on the assignment, from left to right.*

Consider $y' = (2-y)(3-y)$ & $y' = (y-2)(3-y)$. These are both autonomous DE's (& the other two DE's aren't). Therefore, their direction fields will both have constant slope along horizontal strips. A & C have this property (& B does not).
Now, both have constant solutions $y=2$ & $y=3$.
Let's look at their phase portrait:

$y' = (2-y)(3-y)$ 

$y' = (y-2)(3-y)$ 

$\therefore$ $y' = \underline{(2-y)(3-y)}$ corresponds to $\underline{C}$, since in C slope is pos. above $y=3$, neg. below $y=3$ & above $y=2$, & pos. below $y=2$.

$\therefore$ $y' = \underline{(y-2)(3-y)}$ corr. to $\underline{A}$, since in A slope is neg. in $(3,\infty)$, pos. in $(2,3)$, & neg. in $(-\infty,2)$.

Now, $\underline{B}$ is s.t. slopes are constant along vertical strips $\Rightarrow$ y' should only depend on x $\Rightarrow$ B corr. to $y' = \underline{(2-x)(3+x)}$. [Could also see this since horizontal slope at $x=-3$ & $x=2$].

Math 2C03 - Assignment #3 [WeBWorK]

1. Consider the slope field shown.

(a) For the solution that satisfies $y(0)=0$, sketch the solution curve & estimate the following:

$y(1) \approx -0.75$ & $y(-1) \approx 0.75$.

(b) For $y(0)=1$, $y(1) \approx 1$ & $y(-1) \approx 1$.

(c) For $y(0)=-1$, $y(1) \approx -1$ & $y(-1) \approx -1$.

2. Consider the slope field shown.

(a) For $y(0)=0$, $y(1) \approx 0.75$ & $y(-1) \approx 0.5$.

(b) For $y(0)=1$, $y(0.5) \approx 1.75$ & $y(-1) \approx 0.75$

(c) For $y(0)=-1$, $y(1) \approx -2$ & $y(-1) \approx 0$.

3. Match the eq^{n} w/ the direction field. * the order of these will vary b/w users *

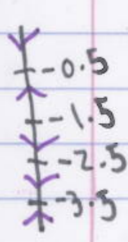
$y' = -\dfrac{(2x+y)}{2y}$ $\leftrightarrow$ A. [when $x=0$, slope is $-1/2$. only one picture has this property].

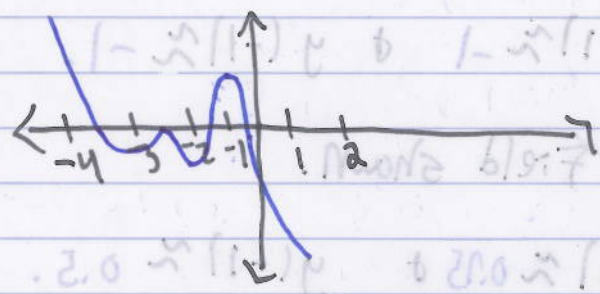
$y' = y + xe^{-x} + 1$ $\leftrightarrow$ C. [when $y=0$, slope will be positive when $x>0$. This brings us to B or C. But in B we have a slope of 0 when $x=0$, which isn't the case here. $\therefore$ C].

$y' = 2xy + 2xe^{-x^2}$ ⟷ B. [When x=0 slope is 0. only B has this property].

$y' = 2\sin(3x) + 1 + y$. ⟷ D [when x = 0, x = −π, x = π we'll have a slope of y + 1. This is the only direction field with that property].

4. Given the DE $x'(t) = f(x(t))$, list the constant solutions in increasing order & indicate if they're stable, semi-stable, or unstable.





−3.5 stable,
−2.5 semi-stable,
−1.5 unstable,
−0.5 stable.

5. Match the DE with the direction field.

$y' = \sin x \sin y$ ⟷ B. [when x = 0 the slope will be zero. B & D have this property, but in D we also have slope zero at y = −1, which isn't the case here. ∴ B]. * answers will vary b/w users *

$y' = x(-1-y)$ ⟷ D. [slope of zero when x=0 & when y=−1].

$y' = x + y - 1$ ⟷ A. [when y=1, positive slope when x>0 & neg. slope when x<0. A only direction field with this property].

$y' = -1 - y$ ⟷ C. [autonomous DE, so for each fixed y the slope is the same ∀x. C only direction field w/ this property].

6. Solve $y' - \frac{2}{x} y = \frac{y^5}{x^7}$, $y(1) = 1$.

Bernoulli's eq'n w/ $n = 5$.

$u = y^{1-5} = y^{-4}$

$\frac{du}{dx} = -4y^{-5} y'$

$\Leftrightarrow -\frac{1}{4}\frac{du}{dx} = y^{-5} y'$

$y^{-5} y' - \frac{2}{x} y^{-4} = \frac{1}{x^7}$

$-\frac{1}{4}\frac{du}{dx} - \frac{2}{x} u = x^{-7}$

$u' + \underbrace{8x^{-1}}_{P(x)} u = \underbrace{-4x^{-7}}_{F(x)}$ linear

$\int P(x)dx = \int 8x^{-1} dx = 8\ln x.$

$u = e^{-8\ln x}\left[\int e^{8\ln x}(-4x^{-7})dx\right]$

$= x^{-8}\left[-4\int x^8 x^{-7} dx\right] = -4x^{-8}\left[\int x\,dx\right] = -4x^{-8}\left[\frac{1}{2}x^2 + c\right]$

$= -2x^{-6} - 4x^{-8}c.$

$\therefore y^{-4} = -2x^{-6} - 4x^{-8}c.$

$y(1) = 1 \Rightarrow 1 = -2 - 4c \Rightarrow 3 = -4c \Rightarrow c = -3/4.$

$\therefore y^{-4} = -2x^{-6} + 3x^{-8} \Leftrightarrow y^{-4} = x^{-6}(-2 + 3x^{-2})$

$\Leftrightarrow y^4 = \frac{x^6}{\frac{3}{x^2} - 2} \Leftrightarrow y^4 = \frac{x^8}{3 - 2x^2} \Leftrightarrow \boxed{y = \frac{x^2}{(3-2x^2)^{1/4}}}.$

choose pos., b/c need $y(1) = 1$.

7. Solve the IVP $\frac{dy}{dx} = \frac{3x\sec\left(\frac{3y}{x}\right) + 9y}{9x}$, $y(1) = \pi/6$.

$$\underbrace{\left(3x\sec\left(\tfrac{3y}{x}\right) + 9y\right)}_{M} dx + \underbrace{(-9x)}_{N} dy = 0.$$

Here M & N are homogeneous of degree 1. Indeed,

$$M(tx,ty) = 3tx\sec\left(\frac{3ty}{tx}\right) + 9ty = tM(x,y).$$

$$N(tx,ty) = -9tx = tN(x,y). \checkmark \quad \therefore \text{ homog. D.E.}$$

$$\left.\begin{array}{l} y = ux \\ dy = x\,du + u\,dx. \end{array}\right\} \left(3x\sec(3u) + 9ux\right)dx - 9x(x\,du + u\,dx) = 0$$

$$\Leftrightarrow 3x\sec(3u)\,dx - 9x^2\,du = 0 \quad \text{separable}$$

$$\Leftrightarrow 3x\sec(3u)\,dx = 9x^2\,du \Leftrightarrow \int \frac{1}{3x}\,dx = \int \frac{1}{\sec(3u)}\,du$$

$$\Leftrightarrow \tfrac{1}{3}\ln x = \int \cos(3u)\,du \Leftrightarrow \tfrac{1}{3}\ln x = \tfrac{1}{3}\sin(3u) + C$$

$$\Leftrightarrow \ln x - c = \sin\left(\frac{3y}{x}\right) \Leftrightarrow \sin^{-1}(\ln x - c) = \frac{3y}{x}$$

$$\Leftrightarrow y = \tfrac{1}{3}x\sin^{-1}(\ln x - c).$$

$$y(1) = \pi/6 \Rightarrow \pi/6 = \tfrac{1}{3}\sin^{-1}(-c) \Leftrightarrow \pi/2 = \sin^{-1}(-c)$$

$$\Leftrightarrow \sin(\pi/2) = -c \Leftrightarrow c = -1.$$

$$\therefore \boxed{y = \tfrac{1}{3}x\sin^{-1}(\ln x + 1)}.$$

8. Solve the IVP $6x\frac{dy}{dx} = 6y + 6\sqrt{25x^2-y^2}$; $y(1) = 5\sin(1)$.

$\underbrace{(6y + 6\sqrt{25x^2-y^2})}_{M}dx \underbrace{-6x}_{N}dy = 0$. This is homog. of deg. 1, since

$M(tx,ty) = 6ty + 6\sqrt{25t^2x^2-t^2y^2} = t(6y + 6\sqrt{25x^2-y^2}) = tM(x,y)$

& $N(tx,ty) = -6xt = tN(x,y)$.

Let $y = ux$, $dy = x\,du + u\,dx$ $\Big\}$ $(6ux + 6\sqrt{25x^2-u^2x^2})dx - 6x(x\,du + u\,dx)$

$\Leftrightarrow 6x\sqrt{25-u^2}\,dx = 6x^2\,du$ ← Separable

$\Leftrightarrow \int \frac{1}{x}dx = \int \frac{1}{\sqrt{25-u^2}}du$ Let $u = 5\sin w$, $du = 5\cos w\,dw$.

$\Leftrightarrow \ln x = \int \frac{5\cos w}{5\sqrt{1-\sin^2 w}}dw$

$\Leftrightarrow \ln x = \int \frac{\cos w}{\cos w}dw \Leftrightarrow \ln x = w + c \Leftrightarrow \ln x = \sin^{-1}(\frac{1}{5}u) + c$.

$\Leftrightarrow \ln x = \sin^{-1}(\frac{y}{5x}) + c$.

$y(1) = 5\sin(1) \Rightarrow 0 = \sin^{-1}(\sin(1)) + c$

$\Rightarrow 0 = 1 + c \Rightarrow c = -1$.

$\therefore \ln x = \sin^{-1}(\frac{y}{5x}) - 1 \Leftrightarrow \boxed{y = 5x\sin(\ln x + 1)}$.

9. Solve the IVP $\frac{dy}{dx} + 4y = e^{-2x} y^5$, $y(1) = \left(\frac{18}{5e^{-2}}\right)^{\frac{1}{4}}$.

Bernoulli's eq'n w/ $n=5$.

let $u = y^{1-5} = y^{-4}$.

$\frac{du}{dx} = -4y^{-5}\frac{dy}{dx}$

$-\frac{1}{4}\frac{du}{dx} = y^{-5}\frac{dy}{dx}$

$y^{-5}\frac{dy}{dx} + 4y^{-4} = e^{-2x}$

$-\frac{1}{4}\frac{du}{dx} + 4u = e^{-2x}$

$\underbrace{u' - 16u}_{P(x)} = \underbrace{-4e^{-2x}}_{F(x)}$ ← linear

$\int P(x)dx = \int -16dx = -16x$.

$u = e^{16x}\left[\int e^{-16x}(-4e^{-2x})dx\right] = -4e^{16x}\left[\int e^{-18x}dx\right]$

$= -4e^{16x}\left[-\frac{1}{18}e^{-18x} + c\right]$

$\Leftrightarrow y^{-4} = \frac{2}{9}e^{-2x} - 4ce^{16x}$. $y(1) = \left(\frac{18}{5e^{-2}}\right)^{\frac{1}{4}}$

$\Rightarrow \left(\frac{18}{5e^{-2}}\right)^{-1} = \frac{2}{9}e^{-2} - 4e^{16}c$

$\frac{18}{4}$ 72

$\Rightarrow \frac{5e^{-2}}{18} - \frac{4}{18}e^{-2} = -4e^{16}c \Rightarrow \frac{e^{-2}}{18}\cdot -\frac{1}{4}e^{-16} = c \Rightarrow c = -\frac{1}{72}e^{-18}$.

$\therefore y^{-4} = \frac{2}{9}e^{-2x} + \frac{1}{18}e^{16x-18}$

$\Rightarrow \boxed{y = \left(\frac{2}{9}e^{-2x} + \frac{1}{18}e^{16x-18}\right)^{-\frac{1}{4}}}$.

10. Solve the IVP $y' = (x+y-3)^2$, $y(0) = 0$.

(a) To solve this, we should make the substitution

$u = x+y-3.$
$u' = 1 + y'.$

(b) After the substitution in the previous part, we obtain the following DE in x, u, u':

$$u' - 1 = u^2.$$

(c) The solution to the IVP is:

$1 + \tan^2 u = \sec^2 u$
$u = \tan w$
$du = \sec^2 w \, dw$

Separable $u' - 1 = u^2$

$\Leftrightarrow \frac{du}{dx} = u^2 + 1 \Leftrightarrow \int \frac{1}{u^2+1} \, du = \int dx$

$\Leftrightarrow \int \frac{\sec^2 w}{\sec^2 w} \, dw = x + c \Leftrightarrow \tan^{-1} u = x + c \Leftrightarrow u = \tan(x+c)$

$\Leftrightarrow x + y - 3 = \tan(x+c) \Leftrightarrow y = -x + 3 + \tan(x+c).$

$y(0) = 0 \Rightarrow 0 = 3 + \tan(c) \Leftrightarrow c = \tan^{-1}(-3).$

$\therefore \boxed{y = -x + 3 + \tan(x + \tan^{-1}(-3))}.$