Math 2C03 - Assignment #2 [Written Part]

[4pts] 1. Consider the IVP $2y' + 8xy = x^3 e^{x^2}$; $y(0) = 2$. Without solving this IVP, explain why a solution exists. Can there exist more than one solution to this IVP on a given interval? Explain.

[From class] Recall: Theorem [Existence & Uniqueness of 1^{st}-order Linear IVPs]

Consider the IVP $y' + P(x)y = F(x)$, $y(x_0) = y_0$.
If $P(x)$ & $F(x)$ are cont. on an interval I containing x_0, then $\exists!$ solution of the IVP on I.

Here, our eq^n is linear 1^{st}-order: $y' + \underbrace{4x}_{P(x)} y = \underbrace{\tfrac{1}{2} x^3 e^{x^2}}_{F(x)}$.

$P(x) = 4x$ is a polynomial, & so is cont. on $(-\infty, \infty)$.

$F(x) = \frac{1}{2} x^3 e^{x^2}$ is also cont. on $(-\infty, \infty)$, since x^3 & e^{x^2} are cont..

$\therefore$ By the Theorem, $\exists!$ solution to the IVP on $(-\infty, \infty)$.

i.e.] A solution exists on $(-\infty, \infty)$ & this solution is unique, so there can't exist more than one solution on any interval, since unique on $(-\infty, \infty)$.

[2pts] 2. Consider the 1^{st}-order DE $(y')^2 + 8 = 0$. Does this eq^n possess any real solutions? i.e. Can there exist a real-valued function $y = \phi(x)$ which satisfies this DE on some interval? Explain.

$(y')^2 + 8 = 0 \Rightarrow \underbrace{(y')^2}_{\geq 0} = \underbrace{-8}_{< 0}$. Suppose $y = \phi(x)$ is a real solution.

Then $y = \phi(x)$ is a real-valued function $\Rightarrow$ $y(x) \in \mathbb{R}\ \forall x$.

This means that $y'(x)$ is also a real-valued function $\Rightarrow (y'(x))^2 \geq 0\ \forall x$.

But then this contradicts $(y')^2 = -8 < 0$.

$\therefore$ No real solution can exist.

[4pts] 3. Suppose you are given a 1^{st}-order DE $y' = f(x,y)$, which satisfies the hypotheses of Theorem 1.2.1 in some rectangular region R. Could 2 different solution curves in its 1-parameter family of solutions intersect at a point in R? Why or why not?

Recall: Theorem 1.2.1: let $R = [a,b] \times [c,d]$ contain (x_0, y_0) in its interior. If $f(x,y)$ & $\frac{\partial f}{\partial y}$ are cont. on $R \Rightarrow \exists$ some interval $I_0 : (x_0 - h, x_0 + h), h > 0$, contained in $[a,b]$, & a unique function $y(x)$, defined in I_0, that is a solution to the IVP $y' = f(x,y)$, $y(x_0) = y_0$.

Theorem 1.2.1 satisfied for a region R $\Rightarrow$ for all points (x_0,y_0) in R there is an interval I_0 containing that point where

the solution to $y' = F(x,y)$, $y(x_0) = y_0$
is unique.

In particular, suppose 2 different solution curves
$G(x,y,c)$ & $G(x,y,\tilde{c})$ intersect at (x_0, y_0).

Restricting these to I_0, we would have two
distinct solution curves going through (x_0, y_0) on
I_0, which is a contradiction.

$\therefore$ $c = \tilde{c}$. i.e., The 2 curves must be the same.

$\therefore$ Two different solution curves can't intersect
at a point (x_0, y_0) in R.

Math 2C03 - Assignment #2 [WeBWork]

1. Find the solution of the DE $(\ln y)^2 y' = x^2 y$ which satisfies the initial condition $y(1) = e^2$.

$u = \ln y$
$du = \frac{1}{y}dy$

$$\int \frac{(\ln y)^2}{y} dy = \int x^2 \, dx$$

$$\int u^2 \, du = \tfrac{1}{3}x^3 + c$$

$$\tfrac{1}{3}(\ln y)^3 = \tfrac{1}{3}x^3 + c$$

$$(\ln y)^3 = x^3 + c$$

$$\ln y = \sqrt[3]{x^3 + c}$$

$$y = e^{\sqrt[3]{x^3 + c}}.$$

$y(1) = e^2 \Rightarrow e^2 = e^{\sqrt[3]{c+1}}$

$\Rightarrow c = 8 - 1 = 7.$

$\therefore \; y = e^{\sqrt[3]{x^3 + 7}}.$

2. A curve passes through the point $(0,5)$ & has the property that the slope of the curve at every pt P is twice the y-coord. of P. What is the eq'n of the curve?

$$\frac{dy}{dx} = 2y$$

$$\int \tfrac{1}{y}\, dy = \int 2 \, dx$$

$$\ln y = 2x + c$$

$$y = e^{2x+c}$$

$$y = c e^{2x}.$$

$y(0) = 5$

$\Rightarrow 5 = c.$

$\therefore \; y = 5e^{2x}.$

3. Find a solution to $\frac{dy}{dx} = xy + 5x + 2y + 10$.

$$\frac{dy}{dx} = (y+5)(x+2)$$

$$\int \frac{1}{y+5}\,dy = \int x+2\,dx$$

$$\ln(y+5) = \tfrac{1}{2}x^2 + 2x + c$$

$$y+5 = c e^{\frac{1}{2}x^2+2x}$$

$$y = c e^{\frac{1}{2}x^2+2x} - 5.$$

They say "a" solution, so choose $y = e^{\frac{1}{2}x^2+2x} - 5$.

[Could choose any constant here].

4. Find a cont. solution for this 1^{st}-order IVP w/ discontinuous RHS:

$$y' + 2ty = \begin{cases} 4t, & 0 \le t \le 6 \\ 0 & 6 < t \end{cases}, \quad y(0) = 3.$$

For $0 \le t \le 6$: $y' + \underbrace{2t}_{P(x)}y = \underbrace{4t}_{F(x)}$

$u = t^2$

$du = 2t\,dt$

$$\int P(x)dx = \int 2t\,dt = t^2.$$

$$y = e^{-t^2}\left[\int e^{t^2} 4t\,dt\right]$$

$$y = e^{-t^2}\left[\int 2e^u\,du\right]$$

$$y = 2e^{-t^2}\left[e^{t^2} + c\right]$$

$$y = 2 + ce^{-t^2}.$$

$y(0) = 3 \Rightarrow$ $3 = 2 + c \Rightarrow c = 1.$

For $t > 6$: $y' + 2ty = 0$

$$y' = -2ty$$

$$\int \tfrac{1}{y}\,dy = \int -2t\,dt$$

$$\ln y = -t^2 + c$$

$$y = ce^{-t^2}.$$

So, $y = \begin{cases} 2 + e^{-t^2}, & 0 \le t \le 6 \\ Ce^{-t^2}, & t > 6 \end{cases}$

satisfies the IVP. To make this cont., need

$2 + e^{-36} = Ce^{-36} \Rightarrow C = 2e^{36} + 1.$

$\therefore y = \begin{cases} 2 + e^{-t^2}, & 0 \le t \le 6 \\ (2e^{36}+1)e^{-t^2}, & t > 6. \end{cases}$

5. Solve the IVP $9(t+1)\frac{dy}{dx} - 5y = 20t$, for $t > -1$ w/ $y(0) = 10$.

$$y' - \underbrace{\frac{5}{9(t+1)}}_{P(x)} y = \underbrace{\frac{20t}{9(t+1)}}_{F(x)}$$

$$\int P(x)dx = -\frac{5}{9}\int \frac{1}{(t+1)}dt = -\tfrac{5}{9}\ln(t+1).$$

$u = t+1$

$du = dt$

$t = u - 1.$

$$y = e^{\frac{5}{9}\ln(t+1)}\left[\int e^{-\frac{5}{9}\ln(t+1)} \frac{20t}{9t+9}dt\right]$$

$$= (t+1)^{5/9}\left[\int (t+1)^{-5/9}\frac{20t}{9(t+1)}dt\right]$$

$$= (t+1)^{5/9}\left[\int u^{-5/9}\frac{20(u-1)}{9u}du\right]$$

$$= (t+1)^{5/9}\frac{20}{9}\left[\int -u^{-14/9} + u^{-5/9}du\right]$$

$$= \frac{20}{9}(t+1)^{5/9}\left[\frac{9}{5}u^{-5/9} + \frac{9}{4}u^{4/9} + c\right]$$

$$= \frac{20}{5} + 5(t+1) + c(t+1)^{5/9}$$

$$\therefore y = 4 + 5(t+1) + c(t+1)^{5/9}.$$

$y(0) = 10$

$\Rightarrow 10 = 4 + 5 + C$

$\Rightarrow C = 1.$

$\therefore y = 5t + 9 + (t+1)^{5/9}.$

6. Solve the IVP $y' + \underbrace{2}_{p(x)}y = \underbrace{30\sin t + 35\cos t}_{f(x)}$, $y(0)=6$.

$u = e^{2x}$
$du = 2e^{2x}$
$v = -\cos t$
$dv = \sin t$

$v = \sin t$
$dv = \cos t$

$\int P(x)\,dx = \int 2\,dx = 2x.$

$$y = e^{-2x}\left[\int e^{2x}(30\sin t + 35\cos t)\,dt\right]$$

$$= e^{-2x}\left[30\left[-e^{2x}\cos t + 2\int \cos t\, e^{2x}\,dt\right] + 35\left[e^{2x}\sin t - 2\int \sin t\, e^{2x}\,dt\right]\right]$$

$$= 30e^{-2x}\underbrace{\left[-e^{2x}\cos t + 2\left[e^{2x}\sin t - 2\int \sin t\, e^{2x}\right]\right]}_{\int e^{2x}\sin t}$$

$$+ 35e^{-2x}\underbrace{\left[e^{2x}\sin t - 2\left[-e^{2x}\cos t + 2\int \cos t\, e^{2t}\right]\right]}_{\int e^{2x}\cos t}$$

$$= \frac{30}{5}e^{-2x}\left[-e^{2x}\cos t + 2e^{2t}\sin t + c\right] + \frac{35}{5}e^{-2x}\left[e^{2x}\sin t + 2e^{2t}\cos t + c\right]$$

$$= -6\cos t + 12\sin t + 7\sin t + 14\cos t + ce^{-2t}$$

$$= 19\sin t + 8\cos t + ce^{-2t}.$$

$y(0) = 6 \Rightarrow 6 = 8 + c \Rightarrow c = -2.$

$\therefore\ y = 19\sin t + 8\cos t - 2e^{-2t}.$

7. Is the DE exact. If so, find F s.t. ∇F is LHS.: $\underbrace{(4xy^2-y)}_{M}dx + \underbrace{(4x^2y-x)}_{N}dy = 0.$

$\frac{\partial M}{\partial y} = 8xy - 1 = \frac{\partial N}{\partial x} \Rightarrow$ exact.

$\nabla F = (M, N) \Rightarrow \frac{\partial F}{\partial x} = M$ & $\frac{\partial F}{\partial y} = N$. $\frac{\partial F}{\partial x} = M \Rightarrow$

$F = \int 4xy^2 - y \, dx = 2x^2y^2 - xy + g(y).$

$\frac{\partial F}{\partial y} = N \Rightarrow 4x^2y - x + g'(y) = 4x^2y - x \Rightarrow g = \int 0 \, dy \Rightarrow g = c.$

$\therefore F = 2x^2y^2 - xy.$

8. Is the DE exact? If so, find F s.t. ∇F = LHS.

$\underbrace{(3e^x \sin y + 4y)}_{M}dx + \underbrace{(4x + 3e^x \cos y)}_{N}dy = 0.$

$\frac{\partial M}{\partial y} = 3e^x \cos y + 4.$ $\frac{\partial N}{\partial x} = 4 + 3e^x \cos y$

$\frac{\partial F}{\partial x} = M \Rightarrow F = \int 3e^x \sin y + 4y \, dx = 3\sin y e^x + 4xy + g(y).$

$\frac{\partial F}{\partial y} = N \Rightarrow 3\cos y e^x + 4x + g'(y) = 4x + 3e^x \cos y \Rightarrow g'(y) = 0 \Rightarrow g(y) = c.$

$\therefore F = 3\sin y e^x + 4xy.$

9. An integrating factor for the DE $(y+2x-1)dx + x(x+y)dy = 0$ is:

ⓐ y, ⓑ $-\frac{2}{y}$, ⓒ y^2, ⓓ $\frac{-2e^y}{y}$, ⓔ ye^y, ⓕ y^2e^y, ⓖ e^y.

Want to find a function, s.t. when both sides of the DE is multiplied by it, the new eq^n will be exact.

ⓐ $(y^2+2xy-y)dx + xy(x+y)dy = 0$

$\frac{\partial M}{\partial y} = 2y+2x-1$. $\frac{\partial N}{\partial x} = 2xy+y^2$. not same.

ⓑ $\left(-2 - 4\frac{x}{y} + \frac{2}{y}\right)dx - \frac{2x}{y}(x+y)dy = 0$.

$\frac{\partial M}{\partial y} = 4\frac{x}{y^2} - \frac{2}{y^2}$. $\frac{\partial N}{\partial x} = -\frac{4x}{y} - 2$. Not same.

$e^y y^{-1}$

$e^y y^{-1} - e^y y^{-2}$

ⓒ $(y^3+2xy^2-y^2)dx + (x^2y^2+xy^3)dy = 0$.

$\frac{\partial M}{\partial y} = 3y^2+4xy-2y$. $\frac{\partial N}{\partial x} = 2xy^2+y^3$. not same.

ⓓ $\left(-2e^y - 4x\frac{e^y}{y} + 2\frac{e^y}{y}\right)dx \; \left(-2x^2\frac{e^y}{y} - 2xe^y\right)dy = 0$.

$\frac{\partial M}{\partial y} = -2e^y + (-4x+2)\left(\frac{e^y}{y} - \frac{e^y}{y^2}\right)$

$\frac{\partial N}{\partial x} = -4x\frac{e^y}{y} - 2e^y$. not same.

ⓔ ye^y. $(y^2e^y+2xye^y-ye^y)dx + (x^2ye^y+xy^2e^y)dy = 0$.

$\frac{\partial M}{\partial y} = 2ye^y + y^2e^y + 2xe^y + 2xye^y - e^y - ye^y = ye^y + y^2e^y - e^y + 2xe^y + 2xye^y$.

$\frac{\partial N}{\partial x} = 2xye^y + y^2e^y$. Not same.

(f) $(y^3e^y + 2xy^2e^y - y^2e^y)\,dx + (x^2y^2e^y + xy^3e^y)\,dy = 0.$

$\frac{\partial M}{\partial y} = 3y^2e^y + y^3e^y + 4xye^y + 2xy^2e^y - 2ye^y - y^2e^y.$

$\frac{\partial N}{\partial x} = 2xy^2e^y + y^3e^y.$ not same.

(g) $(ye^y + 2xe^y - e^y)\,dx + (x^2e^y + xye^y)\,dy = 0.$

$\frac{\partial M}{\partial y} = e^y + ye^y + 2xe^y - e^y = 2xe^y + ye^y.$

$\frac{\partial N}{\partial x} = 2xe^y + ye^y.$ same $\Rightarrow$ exact. $\therefore$ (g).

10. The DE $y + 3y^4 = (y^3 + 3x)y'$ can be written in the form $M\,dx + N\,dy = 0$ where:

$M(x,y) = y + 3y^4$ & $N(x,y) = -y^3 - 3x.$

Use the method of integrating factors to find an implicit solution to the DE:

$\frac{M_y - N_x}{N} = \frac{1 + 12y^3 + 3}{-y^3 - 3x} = \frac{12y^3 + 4}{-(y^3 + 3x)}$ not a function of x alone.

$\frac{N_x - M_y}{M} = \frac{-12y^3 - 4}{y + 3y^4} = \frac{-4(3y^3 + 1)}{y(3y^3 + 1)} = -4/y$ is a function of y alone. ✓

$\therefore e^{\int -4 \frac{1}{y}\,dy} = e^{\ln y^{-4}} = y^{-4}$ is an integrating factor.

$\underbrace{(y^{-3} + 3)}_{\tilde{M}}\,dx + \underbrace{(-y^{-1} - 3xy^{-4})}_{\tilde{N}}\,dy = 0$ is exact.

Indeed $\frac{\partial \tilde{M}}{\partial y} = -3y^{-4} = \frac{\partial \tilde{N}}{\partial x}.$

$\frac{\partial F}{\partial x} = \tilde{M} \Rightarrow F = \int y^{-3} + 3 \, dx = xy^{-3} + 3x + g(y).$

$\frac{\partial F}{\partial y} = \tilde{N} \Rightarrow -3xy^{-4} + g'(y) = -y^{-1} - 3xy^{-4} \Rightarrow g'(y) = -y^{-1}$

$\Rightarrow g = \int -y^{-1} \, dy = -\ln y + c.$

$\therefore F = xy^{-3} + 3x - \ln y.$