Math 1B03/1ZC3 - Tutorial 10

Mar. 21st/25th, 2014

Tutorial Info:

- **Website:** http://ms.mcmaster.ca/∼dedieula.
- **Review Session:** I'll be doing a review session Mon. March 24th, 6:30-8:30pm, HH302. (See Avenue to Learn for additional review sessions.)
- **Math Help Centre:** Wednesdays 2:30-5:30pm.
- **Email:** dedieula@math.mcmaster.ca .

Examples:

- **1.** Let $V = \mathbb{R}^2$ and define addition and scalar multiplication as follows: If $u = (x_1, y_1)$, $v = (x_2, y_2)$, then

$$u + v = \begin{pmatrix} x_1 - 2x_2 + 1 \\ 2y_1 + 3y_2 - 4 \end{pmatrix},$$

$$\alpha u = \begin{pmatrix} \frac{1}{\alpha} x_1 \\ y_1 \alpha^2 \end{pmatrix}.$$

- Is V a vector space with these stated operations? Specify which axioms hold, and which fail.

- **Recall:** A vector space is a set V together with a binary operation "$+$" and a rule for scalar multiplication satisfying 10 axioms. **i.e.** If the axioms hold for all vectors $v, u, w \in V$ and for all scalars $\alpha, \beta \in \mathbb{R}$, then V is a vector space.
- **Note:** Scalars do not have to be in $\mathbb{R}$, but for simplicity I'll use $\mathbb{R}$ here.

Vector Space Axioms:

1. "+" **Closure:** $v, w \in V \Rightarrow v + w \in V$.
2. "+" **Commutativity:** $v, w \in V \Rightarrow v + w = w + v$.
3. "+" **Associativity:** $u, v, w \in V \Rightarrow (u + v) + w = u + (v + w)$.
4. "+" **Identity:** $\exists$ a vector $\bar{0} \in V$, such that $v + \bar{0} = v$, $\forall v \in V$.
5. "+" **Inverse:** For each $v \in V$ $\exists (-v) \in V$ such that $v + (-v) = \bar{0}$.
6. "α" **Closure:** $v \in V \Rightarrow \alpha v \in V \ \forall \alpha \in \mathbb{R}$.
7. "α" **Distributivity:** $\alpha(v + w) = \alpha v + \alpha w \ \forall v, w \in V$, $\alpha \in \mathbb{R}$.
8. **Vector Distributivity:** $(\alpha + \beta)v = \alpha v + \beta v \ \forall v \in V$, $\alpha, \beta \in \mathbb{R}$.
9. "α" **Associativity:** $(\alpha(\beta v) = (\alpha\beta)v \ \forall v \in V$, $\alpha, \beta \in \mathbb{R}$.
10. "α" **Identity:** $1 \times v = v \ \forall v \in V$, $1 \in \mathbb{R}$.

Examples:

- **2.** If $V = \mathbb{R}^2$ is a set with addition and scalar multiplication defined as $u+v = (u_1+v_1+1, u_2+v_2+1), \alpha u = (\alpha u_1, \alpha u_2)$, where $u = (u_1, u_2)$,, $v = (v_1, v_2)$, then what must $\bar{0}$ be?

Examples:

- **3.** Determine which of the following sets are subspaces of P_2 (where P_2 is the vector space of polynomials of degree ≤ 2. e.g. $\{ax^2+bx+c \mid a,b,c \in \mathbb{R}\}$).
- **Recall:** A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and multiplication operations defined on V.
- **Subspace Criterion:** A subset $W \subseteq V$ is a subspace of $V \iff$ the following hold:
  1. W is nonempty.
  2. W is closed under addition (i.e. $u, v \in W \Rightarrow u+v \in W \; \forall$ scalars α).
  3. W is closed under scalar multiplication (i.e. $u \in W \Rightarrow \alpha u \in W \; \forall$ scalars α).
- **a)** $W = \{r(1+x^2) \mid r \in \mathbb{R}\}$.
- **b)** $Y = \{$quadratic polynomials with only real roots$\}$.
- **c)** $Z = \{a+bx \mid a,b \in \mathbb{R}, a^2 = b^2\}$.
- **d)** $J = \{p+qx+rx^2 \mid p,q,r \in \mathbb{R}, r \geq 0\}$.

Examples:

- **4.** Is the set $W_1 = \{(v_1, v_2, 0) |\ v_1, v_2 \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$?

Examples:

- **5.** Consider the following sets of vectors:

$$S_1 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} \right\}, S_2 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\},$$

$$S_3 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

- **a)** Which sets span $\mathbb{R}^3$?
- **Recall:** The **span** of a set $S = \{w_1, \ldots, w_r\}$, is the subspace formed by taking all possible linear combinations of the vectors in S. **i.e.** $\text{span}(S) = \{\alpha_1 w_1 + \ldots \alpha_r w_r | \ \alpha_1, \ldots, \alpha_r \in \mathbb{R}\}$.
- **Recall:** If A is square, then $Ax = b$ is consistent for every $n \times 1$ matrix $b \iff \det(A) \neq 0$.

Examples:

- **b)** Is the vector

$$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \text{ in the span of } S_1?\ S_2?\ S_3?$$

- **c)** Which of these vectors are linearly independent?

- **Recall:** If a set of vectors $S = \{v_1, \ldots, v_r\}$ is such that the equation $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r = \bar{0}$ has only the trivial solution (i.e. $\alpha_1 = \ldots = \alpha_r = 0$), then these vectors are said to be **linearly independent**. If there exist nontrivial solutions, then the vectors are said to be **linearly dependent**.