Math 1B03/1ZC3 - Tutorial 3

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Tutorial Info:

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Elementary Matrices

- An **elementary matrix** is a $n \times n$ matrix that can be obtained from the identity I_n by performing a single elementary row operation.
- **e.g.**

$$E_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

  is an elementary matrix that corresponds to the row operation $r_2 \leftarrow r_2 + r_1$.
- So, when we do a row operation to a $n \times n$ matrix A, this is equivalent to multiplying A by an elementary matrix. **e.g.**

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{r_2 \leftarrow r_2 + r_1} = \begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Examples:

- **1.a)** Consider

$$A = \begin{pmatrix} 2 & -4 \\ -2 & 3 \end{pmatrix}.$$

  Write A as a product of elementary matrices.

- **Recall:** To do this we should:
  1. Reduce A to the identity I.
  2. Keep track of row operations.
  3. Write each row operation as an elementary matrix.
  4. Express the row reduction as matrix multiplication.
  5. Solve for A.

Examples:

- **b)** Is this decomposition of A into elementary matrices unique?

Examples:

- **c)** Find A^{-1} without using the formula

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Examples:

- **Note:** Our work in Question 1 demonstrates why the inverse algorithm works.
- **Inverse Algorithm:** To find the inverse of an invertible matrix A:
  1. Find a sequence of elementary row operations that reduce A to I_n.
  2. Perform those same row operations on I_n to obtain A^{-1}.
- **i.e.** These row operations can be written as elementary matrices: $E_k \dots E_2 E_1 A = I$ $\Rightarrow A^{-1} = E_k \dots E_2 E_1$.
- So, to do this quickly, we perform the row operations represented by $E_k \dots E_1$ simultaneously to A and I_n by adjoining A with I_n: $[A|I_n] \to [I_n|A^{-1}]$.

Examples:

- **2.** Consider

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 5 \\ 3 & 2 & 3 \end{pmatrix}.$$

- Using row operations we could find

$$A^{-1} = \begin{pmatrix} -11 & 1 & 2 \\ 3 & 0 & 1 \\ 9 & -1 & -1 \end{pmatrix}.$$

Examples:

- **a)** Does

$$Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

  have a unique solution?

- **Recall:** We know several equivalent statements, where A is a $n \times n$ matrix:
  (a) A is invertible.
  (b) $Ax = 0$ has only the trivial solution.
  (c) The reduced row echelon form of A is I_n.
  (d) A is expressible as the product of elementary matrices.
  (e) $Ax = b$ is consistent for every $n \times 1$ matrix b.
  (f) $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.

Examples:

- **b)** Solve for *x*.

Examples:

- **3.** Consider

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 7 & 1 \end{pmatrix}.$$

- **a)** Is A invertible?
- **b)** Does $Ax = 0$ have nontrivial solutions?