Math 1B03/1ZC3 - Tutorial 12

Apr. 4th/8th, 2014

Tutorial Info:

- **Website:** http://ms.mcmaster.ca/∼dedieula.
- **Exam Review:** I'll be doing an exam review Mon. Apr. 14th, 2:30-4:30pm in BSB147. (There are also 2 other reviews happening that day. See Avenue for more details.)
- **Math Help Centre:** Wednesdays 2:30-5:30pm.
- **Email:** dedieula@math.mcmaster.ca .

Examples:

- **1. a)** Suppose $x_1 = (1,1,0)$ and $x_2 = (2,2,3)$. Find an orthogonal basis for $\text{span}\{x_1, x_2\}$.

- **Recall:** A set $S = \{v_1, \ldots, v_n\}$ of vectors, where $v_1, \ldots, v_n \in V$ is called a **basis** for V if:
  1. The vectors in S are linearly independent.
  2. S spans V.

- **Gram-Schmidt Process:** To convert a basis $\{u_1, \ldots, u_n\}$ to an orthogonal basis $\{v_1, \ldots, v_n\}$, perform the following computations:
  1. $v_1 = u_1$.
  2. $v_2 = u_2 - \text{proj}_{v_1} u_2$.
  3. $v_3 = u_3 - \text{proj}_{v_1} u_3 - \text{proj}_{v_2} u_3$.
  4. $v_4 = u_4 - \text{proj}_{v_1} u_4 - \text{proj}_{v_2} u_4 - \text{proj}_{v_3} u_4$.

- **b)** Find an orthonormal basis for $\text{span}\{x_1, x_2\}$.

- **Recall:** A set of vectors is called **orthonormal** if it is orthogonal and each vector has norm 1.

Examples:

- **2.** Suppose $x_1 = (1,1,1,1)$, $x_2 = (-1,4,4,1)$, $x_3 = (4,-2,2,0)$, and $\{x_1, x_2, x_3\}$ forms a basis for a subspace of $\mathbb{R}^4$. Find an orthonormal basis for this subspace.

Examples:

- **3.** We know

$$T := \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

  forms a basis for $\mathbb{R}^2$.

- **a)** Find the coordinate vector of $v = (3,5)$ relative to the basis T. i.e. Find $[v]_T$.
- **Recall:** If $S = \{v_1, \ldots, v_n\}$ is a basis for V, and $w = k_1 v_1 + k_2 v_2 + \ldots + k_n v_n$ for $k_1, \ldots, k_n \in \mathbb{R}$, then $[w]_S = (k_1, \ldots, k_n)$ is called the **coordinate vector of** v **relative to** S.
- **b)** Find the vector $w \in \mathbb{R}^2$ whose coordinate vector relative to T is $[w]_T = (4,2)$.

Examples:

- **4.** Suppose x_1, x_2, and x_3 are linearly independent vectors in $\mathbb{R}^3$. Let $W = \text{span}\{x_1, x_2, x_3\}$.
- **a)** What is $\dim(W)$, (i.e. the dimension of W)?
- **Recall:** All bases of a finite dimensional vector space V have the same number of vectors.
- If a finite dimensional vector space V has a basis consisting of n vectors, then by definition, $\dim(V) = n$.

Examples:

- **b)** Let $x_4 \in W$. Is the set $Y = \{x_1, x_2, x_3, x_4\}$ linearly independent?
- **Recall:** Let $\{v_1, \ldots, v_n\}$ be a basis for V. Let S be a set of vectors from V. Then:
  1. If S has $> n$ vectors, then S is linearly dependent.
  2. If S has $< n$ vectors, then S does not span V.
- **c)** Let $x_5, x_6 \in W$. Does $\text{span}\{x_5, x_6\} = W$?

Examples:

- **d)** Which familiar vector space is equal to W?
- **Recall:** Let V be a vector space such that $\dim(V) = n$. Let $S = \{x_1, \ldots, x_n\}$ be a set of vectors in V. Then, S is a basis for $V \Leftrightarrow S$ is linearly independent OR S spans V.
- The **standard basis** for $\mathbb{R}^3$ is $\{(1,0,0),(0,1,0),(0,0,1)\}$.

Examples:

- **5.** Suppose you were given a homogeneous linear system, you solved it, and found solutions: $x = 2s + t - 3r$, $y = 2t$, $z = t$, $w = s$, $u = r$.
- **a)** Find a basis for this solution space.
- **b)** What is the dimension of this solution space?

Examples:

- Suppose A is a 3×4 matrix. Complete the following sentences.

- **a)** The rank of A is at most $\ldots$.
- **b)** The nullity of A is at most $\ldots$.
- **c)** The rank of A^T is at most $\ldots$.
- **d)** The nullity of A^T is at most $\ldots$.

- **Recall:** The **rowspace** of A is the subspace spanned by the rows of A.
- The **columnspace** of A is the subspace spanned by the columns of A.
- The **nullspace** of A is the subspace spanned by the solutions to the equation $Ax = 0$.
- **rank(A)=** $\dim\{\text{columnspace of } A\}$ = $\dim\{\text{rowspace of } A\}$.
- **nullity(A)=** $\dim\{\text{nullspace of } A\}$.
- **Rank-Nullity Theorem:** $\text{rank}(A) + \text{nullity}(A) = n$, where A is a $m \times n$ matrix.

Examples:

- **7.** Suppose A is a 3×5 matrix.
- **a)** Are the rows of A linearly dependent?
- **b)** Are the columns of A linearly dependent?

Examples:

- **8.** Suppose A is a 3×3 matrix whose nullspace is a line through the origin in $\mathbb{R}^3$. Can the row or column space of A be a line through the origin too?