Math 1B03/1ZC3 - Tutorial 10

Mar. 21st/25th, 2014

Tutorial Info:

- **Website:** http://ms.mcmaster.ca/∼dedieula.
- **Review Session:** I'll be doing a review session Mon. March 24th, 6:30-8:30pm, HH302. (See Avenue to Learn for additional review sessions.)
- **Math Help Centre:** Wednesdays 2:30-5:30pm.
- **Email:** dedieula@math.mcmaster.ca .

Examples:

- **1.** Let $V = \mathbb{R}^2$ and define addition and scalar multiplication as follows: If $u = (x_1, y_1)$, $v = (x_2, y_2)$, then

$$u + v = \begin{pmatrix} x_1 - 2x_2 + 1 \\ 2y_1 + 3y_2 - 4 \end{pmatrix},$$

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- **Note:** Scalars do not have to be in $\mathbb{R}$, but for simplicity I'll use $\mathbb{R}$ here.

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10. "α" **Identity:** $1 \times v = v$ $\forall v \in V$, $1 \in \mathbb{R}$.

Examples:

- **2.** If $V = \mathbb{R}^2$ is a set with addition and scalar multiplication defined as $u + v = (u_1 + v_1 + 1, u_2 + v_2 + 1), \alpha u = (\alpha u_1, \alpha u_2)$, where $u = (u_1, u_2)$,, $v = (v_1, v_2)$, then what must $\bar{0}$ be?

Examples:

- **3.** Determine which of the following sets are subspaces of P_2 (where P_2 is the vector space of polynomials of degree ≤ 2. e.g. $\{ax^2+bx+c|\ a,b,c \in \mathbb{R}\}$).

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- **d)** $J = \{p + qx + rx^2 | \ p, q, r \in \mathbb{R}, r \geq 0\}$.

Examples:

- **4.** Is the set $W_1 = \{(v_1, v_2, 0) | \ v_1, v_2 \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$?

Examples:

- **5.** Consider the following sets of vectors:

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- **Recall:** If A is square, then $Ax = b$ is consistent for every $n \times 1$ matrix $b \iff \det(A) \neq 0$.

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- **Recall:** If a set of vectors $S = \{v_1, \ldots, v_r\}$ is such that the equation $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r = \bar{0}$ has only the trivial solution (i.e. $\alpha_1 = \ldots = \alpha_r = 0$), then these vectors are said to be **linearly independent**. If there exist nontrivial solutions, then the vectors are said to be **linearly dependent**.