

Math 1B03/1ZC3 - Tutorial 10



Mar. 21st/25th, 2014

Tutorial Info:

- **Website:** <http://ms.mcmaster.ca/~dedieula>.
- **Review Session:** I'll be doing a review session Mon. March 24th, 6:30-8:30pm, HH302. (See Avenue to Learn for additional review sessions.)
- **Math Help Centre:** Wednesdays 2:30-5:30pm.
- **Email:** dedieula@math.mcmaster.ca .



Examples:

- 1. Let $V = \mathbb{R}^2$ and define addition and scalar multiplication as follows: If $u = (x_1, y_1)$, $v = (x_2, y_2)$, then

$$u + v = \begin{pmatrix} x_1 - 2x_2 + 1 \\ 2y_1 + 3y_2 - 4 \end{pmatrix},$$

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- Note:** Scalars do not have to be in \mathbb{R} , but for simplicity I’ll use \mathbb{R} here.



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10. “ α ” **Identity:** $1 \times v = v \forall v \in V, 1 \in \mathbb{R}$.



Examples:

- 2. If $V = \mathbb{R}^2$ is a set with addition and scalar multiplication defined as $u + v = (u_1 + v_1 + 1, u_2 + v_2 + 1)$, $\alpha u = (\alpha u_1, \alpha u_2)$, where $u = (u_1, u_2)$, $v = (v_1, v_2)$, then what must $\bar{0}$ be?



Examples:

- 3. Determine which of the following sets are subspaces of P_2 (where P_2 is the vector space of polynomials of degree ≤ 2 . e.g. $\{ax^2 + bx + c \mid a, b, c \in \mathbb{R}\}$).



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- **d)** $J = \{p + qx + rx^2 \mid p, q, r \in \mathbb{R}, r \geq 0\}$.



Examples:

- 4. Is the set $W_1 = \{(v_1, v_2, 0) \mid v_1, v_2 \in \mathbb{R}\}$ a subspace of \mathbb{R}^3 ?



Examples:

- 5. Consider the following sets of vectors:

$$S_1 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} \right\}, S_2 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\},$$

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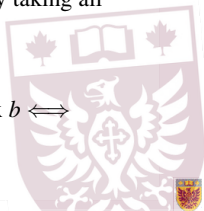
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- **Recall:** If A is square, then $Ax = b$ is consistent for every $n \times 1$ matrix $b \iff \det(A) \neq 0$.



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