Math 1B03/1ZC3 - Tutorial 6

Feb. 11th/ 14th, 2014

Tutorial Info:

- **Website:** http://ms.mcmaster.ca/∼dedieula.
- **Review Session:** I'll be TAing a review session for the midterm on Mon. Feb. 24th, 4:30-6:30pm, in JHE 264. There will also be 4 other reviews happening this day at different times. See Avenue for the times/ rooms.
- **Math Help Centre:** Wednesdays 2:30-5:30pm.
- **Email:** dedieula@math.mcmaster.ca .

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- **b)** Find A^{100}.

- **Recall)** If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$.

Examples:

- **2.** Consider

$$A = \begin{pmatrix} -2 & -27 & 9 \\ 0 & -2 & 0 \\ 0 & -18 & 4 \end{pmatrix}.$$

Find A^k.

Examples:

- **3.** Suppose the population of raccoons in the city in 2010 is 100 and the population of raccoons in the nearby forest is 300. Suppose we also know that 10% of the racoons in the forest move to the city, and 5% of the raccoons in the city move to the forest each year.

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- **Recall:** Our transition matrix takes us from time k to time $k+1$:

$$\begin{pmatrix} w_{k+1} \\ c_{k+1} \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w_k \\ c_k \end{pmatrix}.$$

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- **c)** Does T have a steady-state vector? If so, what is it?
- **Recall:** If P is a regular transition matrix for a Markov chain, then $\exists!$ probability vector q such that $Pq = q$ (i.e. q is an eigenvector corresponding to $\lambda = 1$ and q's entries sum to 1). This vector is called the **steady-state vector**.

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- **e)** How many raccoons will be in the city after 20 years?
- **Recall:** We know $x_n = P^n x_0$, where x_0 is the initial state vector, x_n is the state vector at time n, and P is the transition matrix.