Math 1B03/1ZC3 - Tutorial 3

Jan. 24th/28th, 2014

Tutorial Info:

- **Website:** http://ms.mcmaster.ca/∼dedieula.
- **Math Help Centre:** Wednesdays 2:30-5:30pm.
- **Email:** dedieula@math.mcmaster.ca .

Elementary Matrices

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  is an elementary matrix that corresponds to the row operation $r_2 \leftarrow r_2 + r_1$.
- So, when we do a row operation to a $n \times n$ matrix A, this is equivalent to multiplying A by an elementary matrix. **e.g.**

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{r_2 \leftarrow r_2 + r_1} = \begin{pmatrix} 1 & 2 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

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  4. Express the row reduction as matrix multiplication.
  5. Solve for A.

Examples:

- **b)** Is this decomposition of A into elementary matrices unique?

Examples:

- **c)** Find A^{-1} without using the formula

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

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- So, to do this quickly, we perform the row operations represented by $E_k \dots E_1$ simultaneously to A and I_n by adjoining A with I_n: $[A|I_n] \to [I_n|A^{-1}]$.

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- **2.** Consider

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- Using row operations we could find

$$A^{-1} = \begin{pmatrix} -11 & 1 & 2 \\ 3 & 0 & 1 \\ 9 & -1 & -1 \end{pmatrix}.$$

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  (e) $Ax = b$ is consistent for every $n \times 1$ matrix b.
  (f) $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.

Examples:

- **b)** Solve for *x*.

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- **b)** Does $Ax = 0$ have nontrivial solutions?