Math 1B03/1ZC3 - Tutorial 12

Apr. 4th/8th, 2014

Tutorial Info:

- **Website:** http://ms.mcmaster.ca/∼dedieula.
- **Exam Review:** I'll be doing an exam review Mon. Apr. 14th, 2:30-4:30pm in BSB147. (There are also 2 other reviews happening that day. See Avenue for more details.)
- **Math Help Centre:** Wednesdays 2:30-5:30pm.
- **Email:** dedieula@math.mcmaster.ca .

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- **1. a)** Suppose $x_1 = (1,1,0)$ and $x_2 = (2,2,3)$. Find an orthogonal basis for span$\{x_1, x_2\}$.

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- **Recall:** A set of vectors is called **orthonormal** if it is orthogonal and each vector has norm 1.

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- **2.** Suppose $x_1 = (1,1,1,1)$, $x_2 = (-1,4,4,1)$, $x_3 = (4,-2,2,0)$, and $\{x_1, x_2, x_3\}$ forms a basis for a subspace of $\mathbb{R}^4$. Find an orthonormal basis for this subspace.

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- **Recall:** If $S = \{v_1, \ldots, v_n\}$ is a basis for V, and $w = k_1v_1 + k_2v_2 + \ldots + k_nv_n$ for $k_1, \ldots, k_n \in \mathbb{R}$, then $[w]_S = (k_1, \ldots, k_n)$ is called the **coordinate vector of** v **relative to** S.

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- **b)** Find the vector $w \in \mathbb{R}^2$ whose coordinate vector relative to T is $[w]_T = (4,2)$.

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- **4.** Suppose x_1, x_2, and x_3 are linearly independent vectors in $\mathbb{R}^3$. Let $W = \text{span}\{x_1, x_2, x_3\}$.

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- **Recall:** All bases of a finite dimensional vector space V have the same number of vectors.
- If a finite dimensional vector space V has a basis consisting of n vectors, then by definition, $\dim(V) = n$.

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  1. If S has $> n$ vectors, then S is linearly dependent.

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- **Recall:** Let $\{v_1, \ldots, v_n\}$ be a basis for V. Let S be a set of vectors from V. Then:
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- **c)** Let $x_5, x_6 \in W$. Does $\text{span}\{x_5, x_6\} = W$?

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- The **standard basis** for $\mathbb{R}^3$ is $\{(1,0,0),(0,1,0),(0,0,1)\}$.

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- **b)** What is the dimension of this solution space?

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- **Rank-Nullity Theorem:** $\text{rank}(A) + \text{nullity}(A) = n$, where A is a $m \times n$ matrix.

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- **8.** Suppose A is a 3×3 matrix whose nullspace is a line through the origin in $\mathbb{R}^3$. Can the row or column space of A be a line through the origin too?