Math 1ZC3 - Tutorial #12

1. ⓐ Suppose $x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$. Find an orthogonal basis for span$\{x_1, x_2\}$.

<u>Recall:</u> • To convert a basis $\{u_1, \dots, u_r\}$ to an orthogonal basis $\{v_1, \dots, v_r\}$, perform the following computations:

<u>Gram-Schmidt</u> <u>Process</u>

① $v_1 = u_1$.

② $v_2 = u_2 - \text{proj}_{v_1} u_2$

③ $v_3 = u_3 - \text{proj}_{v_1} u_3 - \text{proj}_{v_2} u_3$

④ $v_4 = u_4 - \text{proj}_{v_1} u_4 - \text{proj}_{v_2} u_4 - \text{proj}_{v_3} u_4$

⋮ etc.

• If $S = \{v_1, \dots, v_n\}$ is a set of orthogonal vectors $\Rightarrow$ S is linearly independent.

We can see that x_1 & x_2 are not multiples of each other, so they are linearly independent $\Rightarrow$ they form a basis for span$\{x_1, x_2\}$. So, we want to use Gram-Schmidt to orthogonalize these vectors.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

$$v_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} - \text{Proj}_{v_1} x_2 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} - \frac{x_2 \cdot v_1}{\|v_1\|^2} v_1 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} - \frac{4}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}.$$

So, $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$ form an orthogonal basis for span$\{x_1, x_2\}$.

<u>Check:</u> $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = 0.$ ✓

(b) Find an orthonormal basis for span$\{x_1, x_2\}$.

<u>Recall:</u> An orthogonal set in which each vector has norm 1 is said to be <u>orthonormal</u>.

So, we must normalize $\begin{pmatrix}1\\1\\0\end{pmatrix}$ & $\begin{pmatrix}0\\0\\3\end{pmatrix}$.

$\left\|\begin{pmatrix}1\\1\\0\end{pmatrix}\right\| = \sqrt{2}$. $\quad \left\|\begin{pmatrix}0\\0\\3\end{pmatrix}\right\| = \sqrt{9} = 3$.

So, $\left\{\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ forms an orthonormal basis for span$\{x_1, x_2\}$.

<u>Check</u> $\left\|\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{pmatrix}\right\| = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1$ ✓ $\quad \left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\right\| = \sqrt{1} = 1$. ✓

2. Suppose $x_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$, $x_2 = \begin{pmatrix}-1\\4\\4\\1\end{pmatrix}$, $x_3 = \begin{pmatrix}4\\-2\\2\\0\end{pmatrix}$, &

$\{x_1, x_2, x_3\}$ forms a basis for a subspace of $\mathbb{R}^4$. Find an orthonormal basis for this subspace.

Again, we will use Gram-Schmidt & normalize:

$\underline{v_1} = \underline{(1,1,1,1)}$. $\quad \underline{v_2} = x_2 - \text{Proj}_{v_1} x_2 = \begin{pmatrix}-1\\4\\4\\1\end{pmatrix} - \dfrac{\begin{pmatrix}-1\\4\\4\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}{4}(1,1,1,1)$

$= (-1,4,4,1) - \frac{8}{4}(1,1,1,1) = \underline{(-3,2,2,-1)}$.

$\underline{v_3} = x_3 - \text{Proj}_{v_1} x_3 - \text{Proj}_{v_2} x_3 = \begin{pmatrix}4\\-2\\2\\0\end{pmatrix} - \dfrac{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}4\\-2\\2\\0\end{pmatrix}}{4}(1,1,1,1) - \dfrac{\begin{pmatrix}-3\\2\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}4\\-2\\2\\0\end{pmatrix}}{9+4+4+1}(-3,2,2,-1)$

$$= \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix} - \frac{4}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{12}{18}\begin{pmatrix} -3 \\ 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1 \\ -1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} -3 \\ 2 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 + 4/3 \\ 1 + 4/3 \\ -1 - 2/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -5/3 \\ 7/3 \\ -5/3 \end{pmatrix}.$$

So, $\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} -3 \\ 2 \\ 2 \\ -1 \end{pmatrix}}_{v_2}, \underbrace{\begin{pmatrix} 1 \\ -5/3 \\ 7/3 \\ -5/3 \end{pmatrix}}_{v_3} \right\}$ forms an orthogonal basis. Now we need to normalize:

27
$4\overline{)108}$
8
28
28
0

$\|v_1\| = \sqrt{4} = 2.$ $\|v_2\| = \sqrt{9+4+4+1} = \sqrt{18} = \sqrt{9 \cdot 2} = 3\sqrt{2}.$

$\|v_3\| = \sqrt{1 + \frac{25}{9} + \frac{49}{9} + \frac{25}{9}} = \sqrt{\frac{108}{9}} = \sqrt{\frac{4 \cdot 27}{9}} = \sqrt{\frac{4 \cdot 3 \cdot \cancel{9}}{\cancel{9}}} = 2\sqrt{3}.$

So, $\left\{ \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 2/3\sqrt{2} \\ 2/3\sqrt{2} \\ -1/3\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/2\sqrt{3} \\ -5/\sqrt{3} \\ 7/\sqrt{3} \\ -5/6\sqrt{3} \end{pmatrix} \right\}$ forms an orthonormal basis.

$$= \left(-1/\sqrt{2}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, \frac{-1}{3\sqrt{2}}\right)$$

3. (see #3 from Tutorial #11 Notes).

4. Suppose x_1, x_2 & x_3 are linearly independent vectors in $\mathbb{R}^3$. Let $W = \text{span}\{x_1, x_2, x_3\}$.

ⓐ What is $\dim(W)$, (i.e. the dimension of W)?

<u>Recall</u>: • All bases of a finite dim'l vector space have the same number of vectors.

- If a finite dim'l vector space V has a basis consisting of n vectors, then by def'n, $\dim(V) = n$.

So, since x_1, x_2, x_3 are linearly independent and they span $W \Rightarrow \{x_1, x_2, x_3\}$ is a basis for W $\Rightarrow \dim(W) = 3$.

(b) Let $x_4 \in W$ $= \text{span}\{x_1, x_2, x_3\}$. Is the set $Y := \{x_1, x_2, x_3, x_4\}$ linearly independent?

Recall: Let $\{v_1, \dots, v_n\}$ be a basis for V. Then:

(a) If $S \subseteq V$ (← subset) has $> n$ vectors $\Rightarrow$ S linearly dependent.
(b) If $S \subseteq V$ has $< n$ vectors $\Rightarrow$ S does not span V.

So, since $\{x_1, x_2, x_3\}$ is a basis for W, $Y \subseteq W$, but Y has $4 > 3$ vectors $\Rightarrow Y$ is not linearly independent.

(c) Let $x_5, x_6 \in W$. Does $\text{span}\{x_5, x_6\} = W$?

$x_5, x_6 \in W$, but
$2 < 3 \Rightarrow \{x_5, x_6\}$ does not span $W \Rightarrow \text{span}\{x_5, x_6\} \neq W$.

(d) Which familiar vector space does W equal?

Recall: • Let V be a vector space s.t. $\dim(V) = n$.
Let $S = \{x_1, \dots, x_n\}$ s.t. $x_1, \dots, x_n \in V$.
Then, S is a basis for $V \iff S$ is linearly independent or S spans V.

- The standard basis for $\mathbb{R}^3$ is $\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$.

Since the standard basis for $\mathbb{R}^3$ has 3 elements $\Rightarrow$ $\dim(\mathbb{R}^3) = 3$.

Let $S = \{x_1, x_2, x_3\}$. We know $x_1, x_2, x_3 \in \mathbb{R}^3$. We also know x_1, x_2, x_3 are 3 linearly independent vectors $\Rightarrow$ S is a basis for $\mathbb{R}^3$ $\Rightarrow$ $\text{span}\{x_1, x_2, x_3\} = \mathbb{R}^3$ $\Rightarrow$ $W = \mathbb{R}^3$.

(We could have also used the Theorem that says: If W is a subspace of V, then $W = V \Leftrightarrow \dim(W) = \dim(V)$).

5. Suppose I gave you a homogeneous system of equations, you solved it, and found:

$x = 2s + t - 3r$
$y = 2t$
$z = t$
$w = s$
$u = r$.

ⓐ What would be a basis for your solution space?

ⓑ What is the dimension of this solution space?

ⓐ We can rewrite this solution as:

$$\begin{pmatrix} x \\ y \\ z \\ w \\ u \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} r.$$

decomposing this way always guarantees that these vectors are linearly independent.

So, $\left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for this homogeneous system (a.k.a. basis for the <u>nullspace</u>).

ⓑ $\therefore$ The dimension of this solution space is 3.