Math 1B03 - Tutorial #9

1. Find a unit vector that has the same direction as $(-4,-3)$.

<u>Recall</u>: a vector of norm 1 is called a unit vector; <u>i.e.</u>, if $\|u\|=1$, then u is a unit vector.

$\|(-4,-3)\| = \sqrt{(-4)^2+(-3)^2} = \sqrt{16+9} = 5.$

So, $u = \frac{1}{5}(-4,-3) = \left(\frac{-4}{5}, \frac{-3}{5}\right)$ is a unit vector with the same direction as $(-4,-3)$.

<u>Check</u> $\left\|\left(\frac{-4}{5}, \frac{-3}{5}\right)\right\| = \sqrt{\left(\frac{-4}{5}\right)^2 + \left(\frac{-3}{5}\right)^2}$

$= \sqrt{\frac{16}{25}+\frac{9}{25}} = \sqrt{1} = 1.$ ✓

2. Let $u=(0,2,2,1)$ & $v=(1,1,1,1)$. Verify that the Cauchy-Schwartz inequality holds.

<u>Recall</u>: <u>Cauchy-Schwartz Inequality</u>: $|u \cdot v| \le \|u\|\|v\|.$

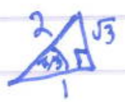
$|u\cdot v| = |0*1+2*1+2*1+1*1| = 5.$

$\|u\|\|v\| = \sqrt{0^2+2^2+2^2+1^2} * \sqrt{1^2+1^2+1^2+1^2}$
$= \sqrt{9} * \sqrt{4} = 3*2 = 6.$

And $5 \le 6.$ ✓

3. Suppose $||u|| = 2$, $||v|| = 1$, & $u \cdot v = 1$. What is the angle in radians between u & v?

Recall: $\cos\theta = \dfrac{u \cdot v}{||u|| \, ||v||}$.



So, $\cos\theta = \frac{1}{2} \rightarrow \boxed{\theta = \pi/3}$.

4. (a) Let $u = (1,0,1)$ & $v = (0,1,1)$. Find a unit vector orthogonal to both u & v.

Recall: Two vectors u & v are orthogonal if $u \cdot v = 0$.

(Note: Could have also used cross product to find this vector. $||u \times v||$ is orthog. to u & v.)

So, we want to find a vector $w = (x,y,z)$ s.t. $u \cdot w = 0$ & $v \cdot w = 0$, & $||w|| = 1$.

$$u \cdot w = \begin{pmatrix}1\\0\\1\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = x + z = 0.$$

$$v \cdot w = \begin{pmatrix}0\\1\\1\end{pmatrix} \cdot \begin{pmatrix}x\\y\\z\end{pmatrix} = y + z = 0.$$

$$\begin{bmatrix}1 & 0 & 1\\0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1 & 0 & 1 & 0\\0 & 1 & 1 & 0\end{array}\right] \qquad \begin{aligned}x &= -z\\ y &= -z\\ z &= t\end{aligned} \qquad \begin{pmatrix}-1\\-1\\1\end{pmatrix} t.$$

$$||(-1,-1,1)|| = \sqrt{(-1)^2 + (-1)^2 + 1^2} = \sqrt{3}.$$

So, $w = \frac{1}{\sqrt{3}}(-1,-1,1)$ is a unit vector orthogonal to both u & v.

(Also, $\frac{1}{\sqrt{3}}(1,1,-1)$ is a unit vector orthogonal to u & v.)

(b) Do $u, v,$ & w form an orthogonal set?

Recall: A nonempty set of vectors in $\mathbb{R}^n$ is called an orthogonal set if all pairs of distinct vectors in the set are orthogonal.

We can see $u \cdot v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 + 0 + 1 = 1 \neq 0.$

So, u & v are not orthogonal $\Rightarrow \{u, v, w\}$ is not an orthogonal set.

5. What does the equation $-2(x+1) + (y-3) - (z+2) = 0$ represent?

Recall: The point normal eq'n of a plane is
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$
where $P_0(x_0, y_0, z_0)$ is a specific point on the plane, $P(x, y, z)$ is an arbitrary point on the plane, and $n = (a, b, c)$ is the normal vector to the plane.

So, our eq'n represents a plane going through the point $(-1, 3, -2)$, with normal $n = (-2, 1, -1)$.

6. Let $u = (6, 2)$ & $a = (3, -9)$.

(a) Find the vector component of u along a.

Recall: If u & a are vectors in $\mathbb{R}^n$ s.t. $a \neq 0$, then we can write $u = w_1 + w_2$, where

$w_1 = \text{proj}_a u := \frac{u \cdot a}{\|a\|^2} a$ (vector component of u along a)
(a.k.a. orthogonal projection of u along a),

and $w_2 = u - w_1 = u - \text{proj}_a u$ (component of u orthogonal to a).

$$w_1 = \text{proj}_a u = \frac{u \cdot a}{\|a\|^2} a = \frac{\binom{6}{2} \cdot \binom{3}{-9}}{3^2 + (-9)^2} (3, -9)$$

$$= \frac{18 - 18}{9 + 81} (3, -9) = \frac{0 * (3, -9)}{90} = (0, 0).$$

So, $(0,0)$ is the vector component of u along a.

(which makes sense, since these 2 vectors are orthogonal).

(b) Find the vector component of u orthogonal to a.

$$w_2 = u - w_1 = (6, 2) - (0, 0) = (6, 2).$$

7. Find the distance between the point $(3, 1, -2)$ and the plane $x + 2y - 2z = 4$.

See Theorem 3.3.4, pg. 149 to see why!

Recall: In $\mathbb{R}^3$, the distance b/w a point $P_0(x_0, y_0, z_0)$ and a plane $ax + by + cz + d = 0$ is: $\dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

$$\frac{|1 * 3 + 2 * 1 + (-2) * (-2) - 4|}{\sqrt{1^2 + 2^2 + (-2)^2}} = \frac{5}{\sqrt{9}} = \frac{5}{3}.$$

So, the distance b/w the point & the plane is $5/3$.

8. Consider 2 points $P(2,3,-2)$ & $Q(7,-4,1)$. Find the point on the line segment containing P & Q that is $3/4$ of the way from P to Q.

Recall: The vector with initial point $P_1(x_1, y_1, z_1)$ & terminal point $P_2(x_2, y_2, z_2)$ is given by the formula: $\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

$$\overrightarrow{PQ} = (7-2, -4-3, 1-(-2)) = (5, -7, 3).$$

$$\tfrac{3}{4}\overrightarrow{PQ} = \left(\tfrac{15}{4}, \tfrac{-21}{4}, \tfrac{9}{4}\right).$$

$$P + \tfrac{3}{4}\overrightarrow{PQ} = \left(\tfrac{8}{4} + \tfrac{15}{4}, \tfrac{12}{4} - \tfrac{21}{4}, \tfrac{-8}{4} + \tfrac{9}{4}\right)$$
$$= \left(\tfrac{23}{4}, \tfrac{-9}{4}, \tfrac{1}{4}\right).$$

line

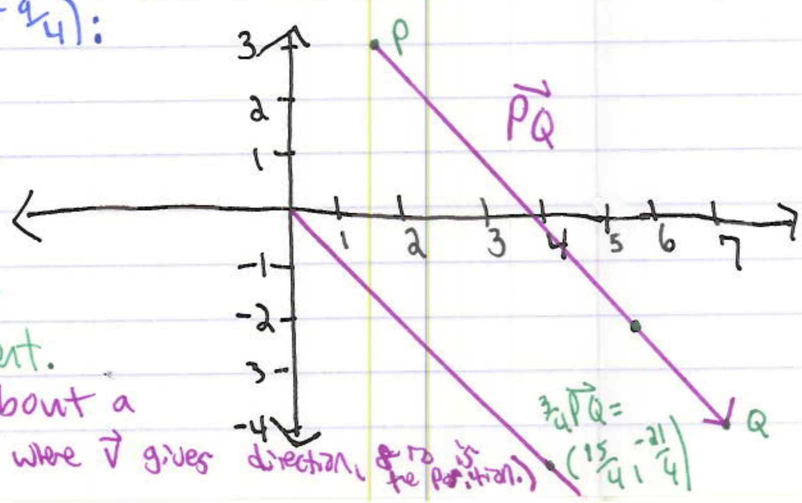
$\boxed{\ell = \vec{r}_0 + t\vec{v}.}$

position vector　　direction vector

So, $Z = \left(\tfrac{23}{4}, \tfrac{-9}{4}, \tfrac{1}{4}\right)$ is the point on $\overrightarrow{PQ}$ $3/4$ of the way from P to Q.

Note: It's easier to see what's happening here in 2 dimensions. If we instead consider $P(2,3)$ & $Q(7,-4)$, then $\tfrac{3}{4}\overrightarrow{PQ} = \left(\tfrac{15}{4}, \tfrac{-21}{4}\right)$, & $P + \tfrac{3}{4}\overrightarrow{PQ} = \left(\tfrac{23}{4}, \tfrac{-9}{4}\right)$:

$\left(\tfrac{15}{4}, \tfrac{-21}{4}\right)$ gives us the right direction, but must add P to it to bring it up to our line segment.



(Recall: We can think about a line as $r_0 + t\vec{v}$, where $\vec{v}$ gives direction, & r_0 is the position.)

9. ⓐ What is the vector equation of the line $4y + 3x = 40$?

<u>Recall:</u> The vector equation of a line ℓ through the point x_0 that is parallel to $\vec{v}$ is $\boxed{\ell = x_0 + t v}$.

position ↑ ↖ direction

$$y = \frac{-3}{4}x + 10$$

rise = y-coord.
run = x-coord.

So, this line has slope $\frac{-3}{4}$ & goes through the point $(0, 10)$, so we have:

$$\boxed{\ell = \begin{pmatrix} 0 \\ 10 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} t}.$$

← not unique.

ⓑ What are the parametric equations of this line?

$$x = 4t$$
$$y = 10 - 3t.$$

ⓒ Which line passes through $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ & is parallel to ℓ?

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} t.$$

↖ direction

<u>Recall:</u> Two lines are parallel if their direction vectors are multiples of each other.

(d) Find a line that is perpendicular to ℓ.

Recall: Two lines are perpendicular if their dot product is 0.

So, we want to find a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ s.t.

$$\begin{pmatrix} 4 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0 \rightarrow 4a - 3b = 0$$

any value of b would have worked... chose $b=8$.

$$\rightarrow 4a = 3b \rightarrow a = \tfrac{3}{4} b. \quad \text{So } \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} \text{ works.}$$

So, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix} t + \begin{pmatrix} 0 \\ 7 \end{pmatrix}$ is perpendicular to ℓ.

this could have been anything... just a position vector.

10. Find a vector equation of the plane in $\mathbb{R}^4$ passing through the point $(2,-1,7,3)$ & parallel to both $(1,0,2,1)$ & $(3,2,4,5)$.

Recall: The eq^n of a plane passing through a point x_0 & parallel to v_1 & v_2 is $\boxed{x = x_0 + v_1 t + v_2 s}$, supposing that v_1 & v_2 are not collinear.

$$\text{So, } \quad x = \begin{pmatrix} 2 \\ -1 \\ 7 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 3 \\ 2 \\ 4 \\ 5 \end{pmatrix} s.$$

11. Find the area of the triangle with vertices $P = (1,1,5)$, $Q = (3,4,3)$, & $R = (1,5,7)$.

Recall: If u & v are vectors in 3-space, then $\|u \times v\|$ = area of the parallelogram determined by u & v.

So, to find the area of our triangular, we want $\frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$.

$\vec{PQ} = (2,3,-2)$. $\vec{PR} = (0,4,2)$.

Recall: $u \times v = \begin{vmatrix} i & j & K \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$, where $\underbrace{i = \begin{pmatrix}1\\0\\0\end{pmatrix}, j = \begin{pmatrix}0\\1\\0\end{pmatrix}, K = \begin{pmatrix}0\\0\\1\end{pmatrix}}_{\text{the standard unit vectors}}$.

So, $\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & K \\ 2 & 3 & -2 \\ 0 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 4 & 2 \end{vmatrix} i - \begin{vmatrix} 2 & -2 \\ 0 & 2 \end{vmatrix} j$

$+ \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} K = 14\underset{\begin{pmatrix}1\\0\\0\end{pmatrix}}{i} - 4\underset{\begin{pmatrix}0\\1\\0\end{pmatrix}}{j} + 8\underset{\begin{pmatrix}0\\0\\1\end{pmatrix}}{K} = (14,-4,8)$.

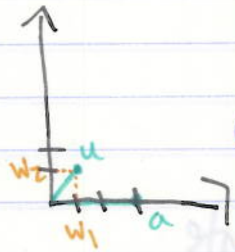
So, $\frac{1}{2}\|\vec{PQ} \times \vec{PR}\| = \frac{1}{2}\sqrt{14^2 + (-4)^2 + 8^2}$

$= \frac{1}{2}\sqrt{276} = \frac{1}{2}\sqrt{4 * 69} = \boxed{\sqrt{69}}$.

12. Let $a = (3,0)$, $u = (1,1)$.

(a) Find the vector component of u along a.

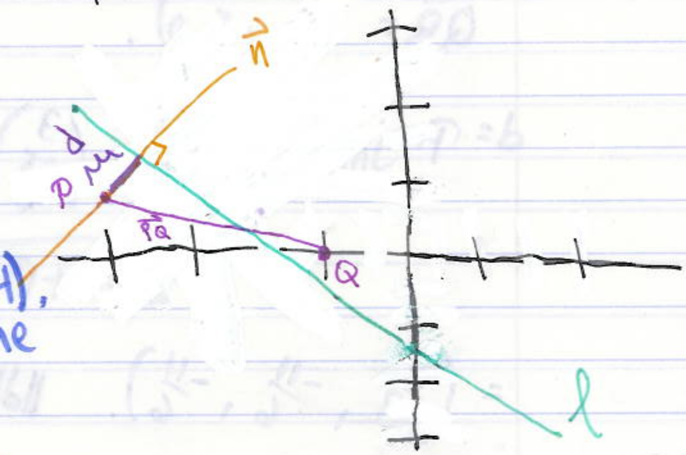
(b) Find the vector component of u orthogonal to a.



(a) $w_1 = \text{proj}_a u = \dfrac{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}3\\0\end{pmatrix}}{3^2+0^2}(3,0) = \dfrac{3}{9}(3,0) = (1,0)$.

(b) $w_2 = u - w_1 = (1,1) - (1,0) = (0,1)$.

13. Find the distance b/w the point $P(-3,1)$ & the line $l: 4x+3y+4=0$.



$3y = -4x - 4$

$y = -\frac{4}{3}x - \frac{4}{3}$.

This line has slope $(3,-4)$. So, the normal to this line is $\vec{n} = (4,3)$.

$Q(-1,0)$ is a point on l, since $4(-1)+3(0)+4=0$. ✓

The line b/w P & Q is $\vec{PQ} = (2,-1)$.

$$d := \text{proj}_{\vec{n}}\vec{PQ} = \frac{\begin{pmatrix}2\\-1\end{pmatrix}\cdot\begin{pmatrix}4\\3\end{pmatrix}}{4^2+3^2}(4,3) = \frac{8-3}{25}(4,3) = \frac{1}{5}(4,3) = \left(\frac{4}{5},\frac{3}{5}\right).$$

$$\|d\| = \sqrt{\frac{4^2+3^2}{5^2}} = \sqrt{\frac{25}{25}} = \boxed{1}.$$

14. Find the distance b/w the parallel planes $P_1 = 2x - y - z = 5$ & $P_2 = -4x + 2y + 2z = 12$.

These planes are parallel since their normal vectors are multiples of each other: $\vec{n}_1 = (2, -1, -1)$, $\vec{n}_2 = (-4, 2, 2)$. $-2\vec{n}_1 = \vec{n}_2$.

arbitrary points

$Q(-3, 0, 0)$ is a point on P_2.

Since the planes are parallel, it suffices to compute the shortest distance b/w Q and P_1:

$R(0, -5, 0)$ is a point on P_1.

$\vec{QR} = (3, -5, 0)$.

$$d = \text{proj}_{\vec{n}_1} \vec{QR} = \frac{\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 0 \end{pmatrix}}{2^2 + 1^2 + 1^2} (2, -1, -1) = \frac{11}{6}(2, -1, -1).$$

$$\|d\| = \frac{11}{6}\sqrt{2^2 + 1 + 1} = \frac{11\sqrt{6}}{6} = \boxed{\frac{11}{\sqrt{6}}}.$$

Suggested Problem — Tues. Mar. 18th /14

Assignment #4

4. In a given year a person may or may not get the flu. Past records show that if a person has the flu one year then there's an 80% chance they won't get the flu the following year. If they don't have the flu in a given year then there's a 25% chance they will get the flu the following year.

(a) If a person has the flu one year, what's the probability they'll also have the flu 2 years later.

First we must set up a transition matrix for this system:

"1" = F
"2" = N

$$P = \begin{array}{c c} & \begin{array}{cc} F & N \end{array} \\ \begin{array}{c} F \\ N \end{array} & \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \end{array}$$

P_{11}: prob. if have flu then get flu next yr = 0.2
P_{12}: prob. if don't have flu then get flu: 0.25
P_{21}: move from flu to no flu: 0.8
P_{22}: no flu → no flu 0.75

$$= \begin{bmatrix} 0.2 & 0.25 \\ 0.8 & 0.75 \end{bmatrix}.$$

At time 0 we have the flu, so

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

year 2 →

$$x_2 = \underbrace{\begin{bmatrix} 0.2 & 0.25 \\ 0.8 & 0.75 \end{bmatrix}^2}_{P^2} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{x_0} = \begin{bmatrix} 1/5 & 1/4 \\ 4/5 & 3/4 \end{bmatrix} \begin{bmatrix} 1/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} \frac{1}{25} + \frac{4}{20} \\ \frac{4}{25} + \frac{12}{20} \end{bmatrix}$$

$$= \begin{bmatrix} 1/25 + 1/5 \\ 4/25 + 3/5 \end{bmatrix} = \begin{bmatrix} 1/25 + \frac{5}{25} \\ \frac{4}{25} + \frac{15}{25} \end{bmatrix} = \begin{bmatrix} 6/25 \\ 19/25 \end{bmatrix} \begin{array}{l} \leftarrow \text{flu} \\ \leftarrow \text{no flu} \end{array}$$

So the probability they'll have the flu 2 years later is $\frac{6}{25}$.

(b) In the long run, what proportion of years does a person have the flu?

Need the steady-state vector (i.e. prob. vector which is an eigenvector w/ eigenvalue $\lambda = 1$).

Reg. stochastic transition matrix $\Rightarrow$ steady-state vector exists.

$$\left[\begin{array}{cc:c} 1/5 - 1 & 1/4 & 0 \\ 4/5 & 3/4 - 1 & 0 \end{array}\right] \quad \left[\begin{array}{cc:c} -4/5 & 1/4 & 0 \\ 4/5 & -1/4 & 0 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 + r_2} \left[\begin{array}{cc:c} 0 & 0 & 0 \\ 4/5 & -1/4 & 0 \end{array}\right]$$

$\frac{4}{5}x = \frac{1}{4}y \Rightarrow x = \frac{5}{16}y = \frac{5}{16}t.$

$y = t$

$$\begin{bmatrix} 5/16 \\ 1 \end{bmatrix} t.$$

$\frac{5}{16}t + t = 1 \Rightarrow \frac{21}{16}t = 1 \Rightarrow t = \frac{16}{21}.$

$$\begin{bmatrix} 5/16 \\ 1 \end{bmatrix} \frac{16}{21} = \underbrace{\begin{bmatrix} 5/21 \\ 16/21 \end{bmatrix}}_{\text{Steady-state vector}} \begin{matrix} \leftarrow \text{flu} \\ \leftarrow \text{no flu} \end{matrix}$$

So, in the long-run $\frac{5}{21}$ is the proportion of years a person will have the flu.