Math 1B03 - Tutorial #7

1. Suppose the population of raccoons in the city in 2010 is 100 & the population of raccoons in the nearby forest is 300. Suppose we also know that 10% of the raccoons move from the woods to the city, & 5% of the raccoons move from the city to the woods, per year.

(a) Set up a transition matrix to describe this phenomenon.

<u>Recall</u>: We should think about each entry a_{ij} in the transition matrix as the probability that the system will move from state j to state i. (pg. 296).

j ← earlier state

i ← later state

Let $W(K)$ denote the pop. of raccoons in the woods & $C(K)$ denote the pop. of raccoons in the city.

Our transition matrix takes us from time K to time $K+1$:

$$\begin{bmatrix} W(K+1) \\ C(K+1) \end{bmatrix} = \begin{array}{c} \\ W \\ C \end{array} \!\!\begin{array}{c} \begin{array}{cc} W & C \end{array} \\ \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \end{array} \begin{bmatrix} W(K) \\ C(K) \end{bmatrix}.$$

So, P_{11}: probability that the raccoon will stay in the woods when it is the woods → $P_{11} = 0.9$.

P_{12}: Prob. that rac. will move from city to woods → $P_{12} = 0.05$.

P_{21}: prob. that rac. will move from woods to city → $P_{21} = 0.1$.

P_{22}: Prob. that raccoons in city will stay in city → $P_{22} = 0.95$. So, $\boxed{\begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix}}$ is our transition matrix.

ⓑ Is T a regular stochastic matrix?

Recall: • A square matrix A is called a stochastic matrix if each of its columns is a probability vector (i.e., the entries of each column add up to 1). (pg. 285).

• A stochastic matrix A is called regular if A, or some positive power of A, has all positive entries.

$$T = \begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix} \text{ is stochastic}$$

since $0.9 + 0.1 = 1$ & $0.05 + 0.95 = 1$ ✓

$\therefore$ T is regular since all of T's entries are positive.

ⓒ Does T have a steady-state vector? If so, what is it?

Recall: If P is a regular transition matrix for a Markov chain, then $\exists!$ probability vector q s.t. $Pq = q$ (i.e., q is an eigenvector corresponding to $\lambda = 1$ & q's entries sum to 1). This vector q is called the steady-state vector. (pg. 288).

So, since T is regular, we know T does have a steady-state vector. Let's find it:

$\underline{\lambda=1}$: $\left[\begin{array}{cc:c} 0.9-1 & 0.05 & 0 \\ 0.1 & 0.95-1 & 0 \end{array}\right]$

$\left[\begin{array}{cc:c} -0.1 & 0.05 & 0 \\ 0.1 & -0.05 & 0 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 + r_2} \left[\begin{array}{cc:c} 0 & 0 & 0 \\ 0.1 & -0.05 & 0 \end{array}\right]$

$\frac{1}{10}x = \frac{1}{20}y \Rightarrow x = \frac{1}{2}t$

$y = t$

So, $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} t$ solves this system.

But, we need that our steady-state vector is an eigenvector corresponding to $\lambda = 1$ <u>AND</u> its entries have to sum to 1.

$\frac{1}{2}t + t = 1 \Leftrightarrow \frac{3}{2}t = 1 \Leftrightarrow t = 2/3.$

So, $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix} 2/3 = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ solves this system

& it's a probability vector. $\therefore \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$ is the steady-state vector.

(d) In the long term, how will the population of raccoons in the city & woods be distributed?

<u>Recall</u>: If q is a steady-state vector for a regular Markov chain, then for any initial probability vector x_0, $\lim_{k \to \infty} P^k x_0 = q$, where P is the transition matrix for this chain. (Theorem 4.12.1 (b) Pg. 288).

$\therefore$ Since our steady-state vector is $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$, $\begin{matrix} \leftarrow \text{woods} \\ \leftarrow \text{city} \end{matrix}$ we know that in the long term, 1/3 of raccoons will be in the woods, & 2/3 of them will be in the city.

ⓔ How many raccoons will be in the city after 20 years?

Recall: We know $X_n = P^n X_0$.

initial state vector

state vector for time n

transition matrix

Here, $P = \begin{bmatrix} 0.9 & 0.05 \\ 0.1 & 0.95 \end{bmatrix}$ & $X_0 = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$.

b/c initially we have 300 raccoons in the woods & 100 in the city.

[Note: If we had some smaller power like A^3, then no need to diagonalize. Just multiply $A^3 X_0$.]

So, we need to find A^{20}. To do this we should diagonalize A, (since $A^{20} = PD^{20}P^{-1}$).

$$\begin{vmatrix} 0.9-\lambda & 0.05 \\ 0.1 & 0.95-\lambda \end{vmatrix} = (0.9-\lambda)(0.95-\lambda) - \frac{1}{10}\cdot\frac{1}{20}$$
$$= \frac{9}{10}\cdot\frac{19}{20} - \frac{9}{10}\lambda - \frac{19}{20}\lambda + \lambda^2 - \frac{1}{200}$$
$$= \lambda^2 - \frac{37}{20}\lambda + \frac{170}{200} = 0$$

I know $\lambda = 1$ is an eigenvalue b/c reg. stochastic, so:

$$\begin{array}{r} \lambda - \frac{17}{20} \\ \lambda - 1 \overline{\big) \lambda^2 - \frac{37}{20}\lambda + \frac{170}{200}} \\ \underline{\lambda^2 - \lambda \qquad\qquad} \\ -\frac{17}{20}\lambda + \frac{170}{200} \\ \underline{-\frac{17}{20}\lambda + \frac{17}{20}} \\ 0 \end{array}$$

So, $\lambda = 1$ & $\lambda = \frac{17}{20}$ are my eigenvalues.

By ⓒ I know $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 1$.

$$\underline{\lambda = \overset{0.85}{\overset{\shortparallel}{17/20}}}: \quad \left[\begin{array}{cc:c} 0.9-0.85 & 0.05 & 0 \\ 0.1 & 0.95-0.85 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc:c} 0.05 & 0.05 & 0 \\ 0.1 & 0.1 & 0 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 - \frac{1}{2} r_2} \left[\begin{array}{cc:c} 0 & 0 & 0 \\ 0.1 & 0.1 & 0 \end{array}\right]$$

$$\tfrac{1}{10} x = -\tfrac{1}{10} y \qquad x = -t$$
$$y = t$$

So $\begin{bmatrix} -1 \\ 1 \end{bmatrix} t$ solves this system $\Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 17/20$.

So, $P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 17/20 \end{bmatrix}$.

$P^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \dfrac{1}{3} = \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix}$.

So, $A = PDP^{-1} \Rightarrow A^{20} = PD^{20}P^{-1}$.

$\therefore X_{20} = A^{20} X_0 = PD^{20}P^{-1} \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$

$$= \underbrace{\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \left(\frac{17}{20}\right)^{20} \end{bmatrix}} \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & -\left(\frac{17}{20}\right)^{20} \\ 2 & \left(\frac{17}{20}\right)^{20} \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix}} \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + \frac{2}{3}\left(\frac{17}{20}\right)^{20} & \frac{1}{3} - \frac{1}{3}\left(\frac{17}{20}\right)^{20} \\ \frac{2}{3} - \frac{2}{3}\left(\frac{17}{20}\right)^{20} & \frac{2}{3} + \frac{1}{3}\left(\frac{17}{20}\right)^{20} \end{bmatrix} \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{12} + \frac{6}{12}\left(\frac{17}{20}\right)^{20} + \frac{1}{12} - \frac{1}{12}\left(\frac{17}{20}\right)^{20} \\ \frac{6}{12} - \frac{6}{12}\left(\frac{17}{20}\right)^{20} + \frac{2}{12} + \frac{1}{12}\left(\frac{17}{20}\right)^{20} \end{bmatrix} \begin{array}{l} \leftarrow \text{\% in woods} \\ \leftarrow \text{\% in city} \end{array}$$

b/c 400 in population.

$\therefore 400 * \left(\frac{6}{12} - \frac{6}{12}\left(\frac{17}{20}\right)^{20} + \frac{2}{12} + \frac{1}{12}\left(\frac{17}{20}\right)^{20}\right)$ raccoons will be in the city after 20 years.

2. Express $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$ as a real number.

$$= \frac{1+2i}{3-4i}\,\frac{(3+4i)}{(3+4i)} + \frac{2-i}{5i}\,\frac{(-5i)}{(-5i)}$$

$i^2=-1$

$$= \frac{3+10i+8i^2}{9-16i^2} + \frac{-10i+5i^2}{-25i^2} = \frac{3+10i-8}{9+16} + \frac{-10i-5}{25}$$

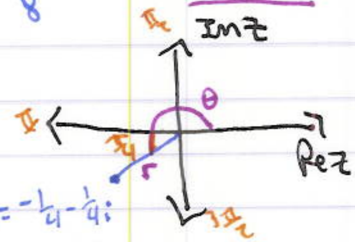
$$= \frac{-5+10i-10i-5}{25} = \frac{-10}{25} = \boxed{\frac{-2}{5}}$$

3. Consider $z = \frac{i}{-2-2i}$.

(a) Express z in rectangular form (i.e., write as $z = a+ib$).

$$= \frac{i}{-2-2i}\,\frac{(-2+2i)}{(-2+2i)} = \frac{-2i-2}{4+4} = \frac{-2-2i}{8} = -\frac{1}{4} - \frac{1}{4}i.$$

(b) Express z in polar form.



$z = r(\cos\theta + i\sin\theta)$
$= re^{i\theta}$

$$r = |z| = \sqrt{\left(-\tfrac{1}{4}\right)^2 + \left(-\tfrac{1}{4}\right)^2} = \sqrt{\tfrac{1}{16}+\tfrac{1}{16}} = \sqrt{\tfrac{2}{16}} = \sqrt{\tfrac{1}{8}} = \frac{1}{\sqrt{4\cdot 2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$z = -\frac{1}{4} - \frac{1}{4}i = \frac{\sqrt{2}}{4}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \frac{\sqrt{2}}{4}\left(\cos\left(\tfrac{5\pi}{4}\right) + i\sin\left(\tfrac{5\pi}{4}\right)\right)$$

$$= \boxed{\frac{\sqrt{2}}{4} e^{\frac{5\pi}{4}i}}.$$

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$\frac{\pi}{4} + \pi = \frac{5\pi}{4}.$

(c) What is Arg z?

<u>Recall:</u> $\arg z = \theta + 2\pi K,\ K \in \mathbb{Z}$ (multivalued), but $-\pi < \operatorname{Arg} z \leq \pi$.

argument

principal argument

$\frac{5\pi}{4} > \pi$. $\frac{5\pi}{4} - 2\pi = \frac{5\pi}{4} - \frac{8\pi}{4} = \frac{-3\pi}{4}$. $-\pi < \frac{-3\pi}{4} \leq \pi$ ✓

So, $\underline{\operatorname{Arg} z} = \underline{\frac{-3\pi}{4}}$.

(d) What is $\bar{z}$?

<u>Recall:</u> If $z = a+bi$, then $\bar{z} = a - bi$.

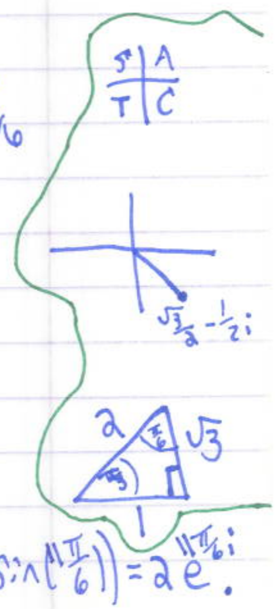
So, $\bar{z} = \frac{-1}{4} + \frac{1}{4}i$.

(Or, in polar form we know that if $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$, since $re^{-i\theta} = r(\cos(-\theta) + i\sin(-\theta))$
$= r(\cos\theta - i\sin\theta) = r\cos\theta - ir\sin\theta = \bar{z}$ ✓).

b/c $\cos\theta$ an even function & $\sin\theta$ an odd function. i.e., $\cos(-\theta) = \cos\theta$ & $\sin(-\theta) = -\sin\theta$.

So, in polar form, $\underline{\bar{z} = \frac{\sqrt{2}}{4}e^{-\frac{5\pi}{4}i}}$.

$2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}$



4. Express $(\sqrt{3} - i)^6$ in polar form.

First, let's express $\sqrt{3} - i$ in polar form.

$r = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$.

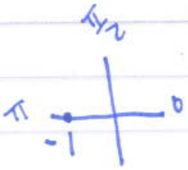
So, $\sqrt{3} - i = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2\left(\cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right)\right) = 2e^{\frac{11\pi}{6}i}$.

So, we want to express $\left(2e^{\frac{11\pi}{6}i}\right)^6$ in polar form.

$\left(2e^{\frac{11\pi}{6}i}\right)^6 = \boxed{2^6 e^{\pi i}}$. since $e^{11\pi i} = e^{\pi i}$ $\left\{ \text{b/c } \begin{array}{l} \cos(11\pi) = \cos(\pi) \\ \sin(11\pi) = \sin(\pi) \end{array} \right)$.

5. Find the solutions to the eq'n $z^3 = -1$.

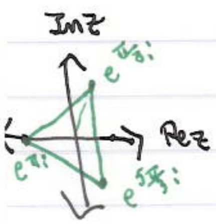
Recall: $z^{\frac{1}{n}} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$,
$k = 0, 1, \dots, n-1$.



First let's ~~right~~ write -1 in polar form:

$r = |-1| = \sqrt{(-1)^2} = 1$. $\theta = \pi$. So, $-1 = \cos\pi + i\sin\pi = e^{\pi i}$.

Now, $z^3 = e^{\pi i}$. Let $z = r_0 e^{\phi i}$.



So, $r_0^3 e^{3\phi i} = e^{\pi i} \to r_0^3 = 1 \to r_0 = 1$
$\to 3\phi = \pi + 2k\pi \to \phi = \frac{\pi}{3} + \frac{2k\pi}{3}$
for $k = 0, 1, 2$.

So, $\phi = \frac{\pi}{3}$, $\frac{\pi}{3} + \frac{2\pi}{3} = \pi$, & $\frac{\pi}{3} + \frac{4\pi}{3} = \frac{5\pi}{3}$.

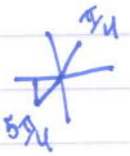
So, $z = e^{\frac{\pi}{3}i}$, $e^{\pi i}$, & $e^{\frac{5\pi}{3}i}$.

6. (a) Find the square roots of $2i$.

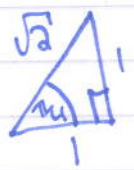
$z = 2i = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) = 2e^{\frac{\pi}{2}i}$.

So, we want to solve $z = \sqrt{2e^{\frac{\pi}{2}i}}$; i.e., $z^2 = 2e^{\frac{\pi}{2}i}$.
Let $z = r_0 e^{\phi i}$.
So, $r_0^2 e^{2\phi i} = 2e^{\frac{\pi}{2}i} \to r_0^2 = 2 \to r_0 = \sqrt{2}$.
$\to 2\phi = \frac{\pi}{2} + 2k\pi \to \phi = \frac{\pi}{4} + k\pi$ for $k = 0, 1$. So, $\phi = \frac{\pi}{4}$ & $\frac{5\pi}{4}$.

So, the square roots of $2i$ are $\underline{\sqrt{2}e^{\frac{\pi}{4}i}}$ & $\underline{\sqrt{2}e^{\frac{5\pi}{4}i}}$.



(b) Express your two roots in rectangular coordinates.



$$\sqrt{2}e^{\frac{\pi}{4}i} = \sqrt{2}\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right)$$

$$= \sqrt{2}\left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right) = \underline{1+i}.$$

$$\sqrt{2}e^{\frac{5\pi}{4}} = \sqrt{2}\left(\cos\left(\tfrac{5\pi}{4}\right) + i\sin\left(\tfrac{5\pi}{4}\right)\right) = \sqrt{2}\left(\tfrac{-1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right)$$

$$= \underline{-1-i}.$$

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