Math 1B03 - Tutorial #6

1. Consider $A = \begin{pmatrix} 3 & 10 \\ 1 & 0 \end{pmatrix}$.

(a) Find a matrix P that diagonalizes A.

<u>Recall</u>: A is said to be <u>diagonalizable</u> if $\exists$ an invertible matrix P s.t. $P^{-1}AP$ is a diagonal matrix. (i.e. $P^{-1}AP = D \Rightarrow A = PDP^{-1}$.) $\leftarrow$ diagonal

<u>Procedure for Diagonalizing a Matrix:</u>

1. Find the eigenvalues $\lambda_1, \dots, \lambda_K$ & eigenvectors $v_1, \dots, v_\ell$ of your matrix A.

2. Create a matrix P by putting your eigenvectors vectors as the columns of P. i.e. $P = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \dots & v_\ell \\ | & | & & | \end{bmatrix}$. (these are not 1's... please let $\begin{matrix} | \\ v_i \\ | \end{matrix}$ denote a column which I'm calling v_i).

3. Create a matrix D by putting your eigenvalues along the diagonal such that the eigenvalue in column i corresponds to the eigenvector in column i of P. i.e. $\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_K \end{bmatrix}$.

4. Find P^{-1}.

5. Check to make sure $A = PDP^{-1}$.

Okay, so let's find A's eigenvalues & eigenvectors:

$$\begin{vmatrix} 3-\lambda & 10 \\ 1 & -\lambda \end{vmatrix} = -\lambda(3-\lambda) - 10 = -3\lambda + \lambda^2 - 10$$
$$= \lambda^2 - 3\lambda - 10 = (\lambda-5)(\lambda+2) = 0$$
$$\lambda_1 = 5 \quad \lambda_2 = -2.$$

Want to find eigenvectors, so solve $(A-\lambda I)\vec{x} = 0$

$\underline{\lambda_1 = 5}$: $\left[\begin{array}{cc:c} 3-5 & 10 & 0 \\ 1 & -5 & 0 \end{array}\right]$ $\left[\begin{array}{cc:c} -2 & 10 & 0 \\ 1 & -5 & 0 \end{array}\right] r_1 \leftarrow r_1 + 2r_2$

$\left[\begin{array}{cc:c} 0 & 0 & 0 \\ 1 & -5 & 0 \end{array}\right]$ $\begin{matrix} x = 5y \\ y = t \end{matrix} \rightarrow x = 5t$

So, $\begin{pmatrix} 5 \\ 1 \end{pmatrix} t$ solves this system of eq^{n} is
$\rightarrow \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is the eigenvector corresponding to $\lambda_1 = 5$.

$\underline{\lambda_2 = -2}$: $\left[\begin{array}{cc:c} 3+2 & 10 & 0 \\ 1 & 2 & 0 \end{array}\right]$ $\left[\begin{array}{cc:c} 5 & 10 & 0 \\ 1 & 2 & 0 \end{array}\right] r_1 \leftarrow r_1 - 5r_2$

$\left[\begin{array}{cc:c} 0 & 0 & 0 \\ 1 & 2 & 0 \end{array}\right]$ $\begin{matrix} x = -2y \\ y = t \end{matrix} \rightarrow x = -2t$

So, $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is the eigenvector corresponding to $\lambda_2 = -2$.

Now, we can form our matrices P & D:

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \quad P = \begin{bmatrix} 5 & -2 \\ 1 & 1 \end{bmatrix}.$$

$$P^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix} \frac{1}{5+2} = \begin{bmatrix} 1/7 & 2/7 \\ -1/7 & 5/7 \end{bmatrix}.$$

$$PDP^{-1} = \begin{bmatrix} 5 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1/7 & 2/7 \\ -1/7 & 5/7 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 4 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 1/7 & 2/7 \\ -1/7 & 5/7 \end{bmatrix} = \begin{bmatrix} 3 & 10 \\ 1 & 0 \end{bmatrix} = A. \checkmark$$

So, $P = \begin{bmatrix} 5 & -2 \\ 1 & 1 \end{bmatrix}$ diagonalizes A.

(b) Find A^{100}.

<u>Recall</u>: $A^K = PD^KP^{-1}$, (if $A = PDP^{-1}$).

Indeed, $A^2 = AA = P\underbrace{DP^{-1}P}_{I}DP^{-1} = PDDP^{-1} = PD^2P^{-1}$, ... etc.,

So, $$A^{100} = \begin{bmatrix} 5 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5^{100} & 0 \\ 0 & (-2)^{100} \end{bmatrix}\begin{bmatrix} 1/7 & 2/7 \\ -\frac{1}{7} & \frac{5}{7} \end{bmatrix}$$

$$= \begin{bmatrix} 5^{101} & (-2)^{101} \\ 5^{100} & (-2)^{100} \end{bmatrix}\begin{bmatrix} 1/7 & 2/7 \\ -1/7 & 5/7 \end{bmatrix} = \begin{bmatrix} \frac{5^{101}}{7} - \frac{(-2)^{101}}{7} & \frac{2\cdot 5^{101}}{7} + \frac{5\cdot(-2)^{101}}{7} \\ \frac{5^{100} - (-2)^{100}}{7} & \frac{2\cdot 5^{100}}{7} + \frac{5\cdot(-2)^{100}}{7} \end{bmatrix}.$$

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2. Consider $A = \begin{bmatrix} -2 & -27 & 9 \\ 0 & -2 & 0 \\ 0 & -18 & 4 \end{bmatrix}$. Find A^K.

$$\begin{vmatrix} -2-\lambda & -27 & 9 \\ 0 & -2-\lambda & 0 \\ 0 & -18 & 4-\lambda \end{vmatrix} = (-2-\lambda)\begin{vmatrix} -2-\lambda & 0 \\ -18 & 4-\lambda \end{vmatrix}$$

$$= (-2-\lambda)(-2-\lambda)(4-\lambda)$$

$$= -(2+\lambda)(-1)(2+\lambda)(4-\lambda)$$

So, $\lambda_1 = -2$ & $\lambda_2 = 4$ are eigenvalues.

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 - 3r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & -3 & 1 \end{array}\right]$$

$r_3 \leftarrow r_3 * \frac{1}{2}$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 - 3r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{9}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$

So, $P^{-1} = \begin{bmatrix} 1 & \frac{9}{2} & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$

<u>Check</u> $PP^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{9}{2} & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

So, $A = PDP^{-1}$

$\Rightarrow A^K = PD^KP^{-1}$

$$\Rightarrow A^K = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} (-2)^K & 0 & 0 \\ 0 & (-2)^K & 0 \\ 0 & 0 & 4^K \end{bmatrix} \begin{bmatrix} 1 & \frac{9}{2} & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}.$$

(a) Are the following 3x3 matrices diagonalizable?

(i) A has eigenvalues $-1, 2, \&\ 4$. ↳ Yes!

(ii) B has eigenvalues $1 \ \& \ 2$. ↳ Maybe... not enough info.

(iii) C has eigenvalues $0 \ \& \ 1$. $\lambda_1 = 0$ has 2 eigenvectors $v_1 = [1, 2, 3]$ & $v_2 = [1, 5, 2]$. & $\lambda_2 = 1$ has eigenvector $[1, 1, 1]$. ↳ Yes! 0 double root, but has 2 "distinct" eigenvectors.

(iv) D has eigenvalues $5 \ \& \ -2$ with eigenvectors v_1 & v_2, respectively. ↳ No! Need 3 distinct eigenvectors.

(v) E has an eigenvalue $\lambda = 0$ with 3 distinct eigenvectors v_1, v_2, v_3. i.e. $\{v_1, v_2, v_3\}$ is a basis for the eigenspace corresponding to $\lambda = 0$.
↳ Yes! B/c 3 distinct eigenvectors.

(b) Which of these matrices are invertible?

Recall: A invertible $\Leftrightarrow$ $\lambda = 0$ is not an eigenvalue of A.

So, A, B, & D are invertible.

C & E are not invertible.

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- If $A = PDP^{-1}$, then A & D have the same determinant. A invertible $\Leftrightarrow$ D is invertible. A & D have the same eigenvalues.