Math 1B03 - Tutorial #5

Review:

Recall: (a) Two matrices are said to **commute** if $AB = BA$.

(b) $\det(\alpha A) = \alpha^n \det(A)$ if $\alpha \in \mathbb{R}$, $A_{n\times n}$.

e.g. → If A is a 7×7 matrix, & $\det(A) = 2$, then $\det(-A) = \det(-1 * A) = (-1)^7 \det(A) = -\det(A) = -2$.

(c) The determinant of a triangular matrix is easy to find. (Triangular means upper triangular, lower triangular, or diagonal).

e.g. → $\det\left(\begin{pmatrix} 1 & 5 & 7 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}\right) \overset{0+0+}{=} 3\begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} = 3(2*1-0) = \underbrace{3*2*1}_{\text{the diagonal entries}} = 6.$

This is true in general... if A triangular, then $\det(A)$ is the product of the diagonal entries.

(d) $\det(AB) = \det(A)\det(B)$.

e.g. → $\det(A^2) = \det(A)\det(A)$.

(e) $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) \rightarrow \operatorname{adj}(A) = \det(A)A^{-1}$.

(f) $\det(A+B) \neq \det(A) + \det(B)$.

(g) A symmetric if $A = A^T$. A skew-symmetric if $A = -A^T$.

e.g. → Suppose A & B skew-symmetric. $(A+B)^T = A^T + B^T \overset{A\,\&\,B \text{ skew-symmetric}}{=} -A - B = -(A+B) \rightarrow (A+B)$ skew-symmetric.

1. Consider $A = \begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix}$. (a) What are the eigenvalues of A?

<u>Recall</u>: • If A is square, then $\vec{x} \in \mathbb{R}^n$ s.t. $\vec{x} \neq \vec{0}$ is called an <u>eigenvector</u> of A if $\boxed{A\vec{x} = \lambda \vec{x}}$ for some $\lambda \in \mathbb{R}$. (i.e., $A\vec{x}$ is a scalar multiple of $\vec{x}$). The scalar λ is called an <u>eigenvalue</u> of A, & $\vec{x}$ is λ's corresponding eigenvector.

• $A\vec{x} = \lambda \vec{x} \iff A\vec{x} - \lambda\vec{x} = 0 \iff (A - \lambda I)\vec{x} = 0$.

by our list of equivalent statements $\Big\{$ We know $\det(A) = 0 \iff A$ not invertible $\iff A\vec{x} = 0$ has non-trivial solutions.

So, since we're looking for vectors $\vec{x}$ s.t. $(A - \lambda I)\vec{x} = 0$, & we know that $\vec{x} \neq 0$ by def^{n}, then by our equivalent statements, that must mean that $\det(A - \lambda I) = 0$.

So, λ is an <u>eigenvalue</u> of $A \iff$ it satisfies the eq^{n} $\boxed{\det(A - \lambda I) = 0.}$ called the <u>characteristic eq^{n}</u>

So, $\det(A - \lambda I) = \begin{vmatrix} 8-\lambda & 9 \\ -6 & -7-\lambda \end{vmatrix} = (8-\lambda)(-7-\lambda) + 54$

$= -56 - 8\lambda + 7\lambda + \lambda^2 + 54 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$

$\iff \lambda = 2$ or $\lambda = -1$.

So, the eigenvalues of A are 2 & -1.

(b) Find all eigenvectors of A.

We know $\vec{x} \neq 0$ is an eigenvector if $A\vec{x} = \lambda\vec{x}$ for some $\lambda \in \mathbb{R}$, or equivalently, if $(A - \lambda I)\vec{x} = 0$.
We already know that $\lambda = 2$ & $\lambda = -1$ are eigenvalues.

$\underline{\lambda=-1}$: $0 = (A - \lambda I)\vec{x} = \left(\begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\vec{x} = \begin{bmatrix} 9 & 9 \\ -6 & -6 \end{bmatrix}\vec{x} = 0.$

let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. So, we want to solve:

$$\begin{bmatrix} 9 & 9 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc:c} 9 & 9 & 0 \\ -6 & -6 & 0 \end{array}\right] \begin{matrix} r_1 \leftarrow r_1 * \frac{1}{9} \\ r_2 \leftarrow r_2 * -\frac{1}{6} \end{matrix} \quad \left[\begin{array}{cc:c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] r_2 \leftarrow r_2 - r_1$$

$$\left[\begin{array}{cc:c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} x = -y = -t \\ y = t. \end{matrix}$$

So, $\begin{bmatrix} -1 \\ 1 \end{bmatrix} t$ is a solution to this eq'n. So, the eigenvectors of A corresponding to $\lambda = -1$ are $\begin{bmatrix} -1 \\ 1 \end{bmatrix} t$ for any $t \in \mathbb{R}$.

(We say $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is a "basis" for the eigenspace corresponding to $\lambda = -1$.) (don't worry about this yet).

$\underline{\lambda=2}$: Similarly: $\begin{bmatrix} 8-\lambda & 9 \\ -6 & -7-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{cc:c} 8-2 & 9 & 0 \\ -6 & -7-2 & 0 \end{array}\right] \quad \left[\begin{array}{cc:c} 6 & 9 & 0 \\ -6 & -9 & 0 \end{array}\right] r_2 \leftarrow r_2 + r_1 \quad \left[\begin{array}{cc:c} 6 & 9 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$6x = -9y \Rightarrow x = \frac{-9}{6}t = \frac{-3}{2}t$$
$$y = t$$

So, $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix} t$ are the set of eigenvectors corresponding to $\lambda = 2$. $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$ is a basis for the eigenspace corresponding to $\lambda = 2$.

don't worry about what this means yet!

Note: Since $t \in \mathbb{R}$, we know if $t = 2$, then $\begin{bmatrix} -3/2 \\ 1 \end{bmatrix} * 2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ is an eigenvector.

$\begin{bmatrix} -3/2 \\ 1 \end{bmatrix} t$ gives us an eigenvector for any value of t.

Do our answers satisfy $A\vec{x} = \lambda\vec{x}$?

Check
$$\underbrace{\begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{x}}$$

using any eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix} t$ works here

e.g. $\begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -16+18 \\ 12-14 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} = -1 * \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ ✓

$$= \begin{bmatrix} -8 + 9 \\ 6 - 7 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underbrace{-1}_{\lambda} * \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{x}}. \checkmark$$

$$\underbrace{\begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} -24+18 \\ 18-14 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \end{bmatrix} = \underbrace{2}_{\lambda} \underbrace{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}_{\vec{x}}. \checkmark$$

2. ⓐ Find all eigenvalues of $A = \begin{bmatrix} 3 & 6 & -6 \\ -1 & -4 & 5 \\ 2 & 2 & -1 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 6 & -6 \\ -1 & -4-\lambda & 5 \\ 2 & 2 & -1-\lambda \end{vmatrix} \quad c_3 \leftarrow c_3 + c_2$$

doing a few row/column ops will make our lives easier

adding a multiple of a row/column to another does not change the determinant.

$$= \begin{vmatrix} 3-\lambda & 6 & 0 \\ -1 & -4-\lambda & -\lambda+1 \\ 2 & 2 & -\lambda+1 \end{vmatrix} \quad r_3 \leftarrow r_3 - r_2 \quad = \begin{vmatrix} 3-\lambda & 6 & 0 \\ -1 & -4-\lambda & -\lambda+1 \\ 3 & \lambda+6 & 0 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \quad = -(-\lambda+1) \begin{vmatrix} 3-\lambda & 6 \\ 3 & \lambda+6 \end{vmatrix} = (\lambda-1)\left[(3-\lambda)(\lambda+6) - 18\right]$$

$$= (\lambda-1)\left[3\lambda + 18 - \lambda^2 - 6\lambda - 18\right] = (\lambda-1)(-\lambda^2 - 3\lambda)$$

$= -\lambda(\lambda-1)(\lambda+3)$. So, the eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -3$.

ⓑ Find the eigenvectors corresponding to $\lambda_3 = -3$.

$$\left[\begin{array}{ccc|c} 3+3 & 6 & -6 & 0 \\ -1 & -4+3 & 5 & 0 \\ 2 & 2 & -1+3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 6 & 6 & -6 & 0 \\ -1 & -1 & 5 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \quad \begin{array}{l} r_1 \leftarrow r_1 + 6r_2 \\ r_3 \leftarrow r_3 + 2r_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 24 & 0 \\ -1 & -1 & 5 & 0 \\ 0 & 0 & 12 & 0 \end{array}\right] \quad \begin{array}{l} r_1 \leftarrow r_1 - 2r_3 \\ r_2 \leftarrow r_2 - \frac{5}{12} r_3 \end{array} \qquad \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 12 & 0 \end{array}\right]$$

$12z = 0 \rightarrow z$

$-x = y \rightarrow x$

$y = t$

So, $\begin{bmatrix}-1\\1\\0\end{bmatrix}t$ are eigenvectors corresponding to

$\lambda_3 = -3$. In particular, $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ is an eigenvector

of A corresponding to $\lambda_3 = -3$.

<u>Check</u>

$$\underbrace{\begin{bmatrix}3 & 6 & -6\\-1 & -4 & 5\\2 & 2 & -1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}-1\\1\\0\end{bmatrix}}_{\vec{x}} = \begin{bmatrix}-3+6\\1-4\\-2+2\end{bmatrix} = \begin{bmatrix}3\\-3\\0\end{bmatrix} = \underbrace{-3}_{\lambda}\cdot\underbrace{\begin{bmatrix}-1\\1\\0\end{bmatrix}}_{\vec{x}}.$$

Ⓒ Is A invertible?

<u>Recall</u>: A invertible $\Leftrightarrow$ $\lambda = 0$ is <u>not</u> an eigenvalue of A.

i.e., A <u>not</u> invertible $\Leftrightarrow$ $\lambda = 0$ <u>is</u> an eigenvalue of A.

So, since we showed $\lambda = 0$ is an eigenvalue $\Rightarrow$ A <u>not</u> invertible.

b/c "$\Rightarrow$" A not invertible $\Rightarrow$ $A\vec{x} = 0$ has nontrivial solutions

$\Rightarrow \exists\ \vec{x} \neq 0$ s.t. $A\vec{x} = 0 \Rightarrow A\vec{x} = 0\vec{x} \Rightarrow 0$ eigenvalue

w/ eigenvector $\vec{x}$.

"$\Leftarrow$" $\lambda = 0$ eigenvalue $\Rightarrow A\vec{x} = 0\vec{x}$ for $\vec{x} \neq 0$ eigenvector

$\Rightarrow A\vec{x} = 0$ has nontrivial solution $\Rightarrow$ A <u>not</u> invertible.

3. (a) Find the eigenvalues of $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) = (2-\lambda)^2$

$\rightarrow$ $\lambda = 2$ is an eigenvalue.

(b) Find all eigenvectors of A.

$(A - \lambda I)\vec{x} = 0$

2 rows of zeros $\rightarrow$ 2 free parameters

$\left[\begin{array}{cc:c} 2-2 & 0 & 0 \\ 0 & 2-2 & 0 \end{array}\right]$ $\left[\begin{array}{cc:c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $x = t$
$y = s$.

So $\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} s$ is the set of all solutions

$\rightarrow$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors $\rightarrow$ the basis for the eigenspace is $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

don't worry about this yet.

Check

$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underbrace{2}_{\lambda} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{x}}$ ✓ $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ✓

Note: Plugging any value of t & s into $\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} s$ will work:

e.g. $t = 4$, $s = 1$ gives us $\begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} 8 \\ 2 \end{bmatrix} = \underbrace{2}_{\lambda} * \underbrace{\begin{bmatrix} 4 \\ 1 \end{bmatrix}}_{\vec{x}}$. ✓

4. Consider $A = \begin{bmatrix} 5 & -3 \\ a & b \end{bmatrix}$ and suppose $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A. What must the eigenvalue λ corresponding to $\vec{x}$ be?

By def^{n} $\vec{x} \neq 0$ is an eigenvector of A if

$$\boxed{A\vec{x} = \lambda \vec{x}} \quad \text{For some } \lambda \in \mathbb{R}.$$

$$A\vec{x} = \begin{bmatrix} 5 & -3 \\ a & b \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ a+b \end{bmatrix} = \underset{\lambda}{2} \underbrace{\begin{bmatrix} 1 \\ \frac{1}{2}(a+b) \end{bmatrix}}_{\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$

Since we know $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector, we need to get that "1" in the top row. We need to factor out a 2 to do this, so $\lambda = 2$ must be the eigenvalue. (It doesn't matter what a & b are ... looking at the first row suffices).

5. (a) Find the eigenvalues of A^{10} if $A = \begin{bmatrix} 8 & 9 \\ -6 & -7 \end{bmatrix}$.
(b) Find the eigenvectors of A^{10}.

From #1, we know $\lambda = 2$ is an eigenvalue of A w/ corresponding eigenvectors $\begin{bmatrix} -3 \\ 2 \end{bmatrix} t$, & $\lambda = -1$ is an eigenvalue w/ eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix} t$.

<u>Recall</u>: If λ is an eigenvalue of A w/ corresponding eigen~~value~~ vector $\vec{x}$, then λ^k an eigenvalue of A^k w/ corresponding eigenvector $\vec{x}$.

<u>e.g.</u>: We know $A\vec{x} = \lambda\vec{x} \to A^3\vec{x} = AA(A\vec{x}) = AA\lambda\vec{x} = \lambda A(A\vec{x}) = \lambda A\lambda\vec{x} = \lambda^2 A\vec{x} = \lambda^3\vec{x}$.

So $(-1)^{10} = 1$ is an eigenvalue of A^{10} w/ eigenvectors $\begin{bmatrix}-1\\1\end{bmatrix}t$

and $2^{10} = 1024$ is an eigenvalue of A^{10} w/ eigenvectors $\begin{bmatrix}-3\\2\end{bmatrix}t$.