Math 1B03 - Tutorial #4

1. Consider $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 5 & 3 \end{bmatrix}$.

Recall: For a 2×2 matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(B) = ad - bc$.

Recall: If D is square, then the minor of entry a_{ij}, M_{ij}, is the determinant of the submatrix that remains after the ith row & jth column are deleted from D.

Cofactor of entry a_{ij}, C_{ij}: is $K M_{ij}$, where $K = 1$ or -1 in accordance with the pattern in the checkerboard array:

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ & & & \ddots & \text{etc.} & \end{bmatrix}.$$

ⓐ Find M_{11}, M_{12}, M_{13}, C_{11}, C_{12} & C_{13}.

$M_{11} = \det\left(\begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}\right) = 3 - 10 = -7.$ $\quad C_{11} = -7.$

$M_{12} = \det\left(\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\right) = 3 - 2 = 1.$ $\quad C_{12} = -1.$

$M_{13} = \det\left(\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix}\right) = 5 - 1 = 4.$ $\quad C_{13} = 4.$

(b) Find det(A).

Recall: You can find det(A) by multiplying the entries in any row or column by their corresponding cofactor & adding the resulting products.

So, $\det(A) = 3C_{11} + 2C_{12} + 4C_{13}$
$= 3*-7 + 2*-1 + 4*4$
$= -21 - 2 + 16 = \boxed{-7}.$

cofactor expansion along first row.

Note: We could have chosen a different row or column.

e.g., $4*C_{13} + 2*C_{23} + 3*C_{33}$

cofactor expansion along 3rd column.

$= 4\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} - 2\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} + 3\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix}$

$= 4(5-1) - 2(15-2) + 3(3-2)$
$= 16 - 26 + 3 = -7.$

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2. Find $\det\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}\right)$.

$\det\left(\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}\right) = 2*\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2(2-1) = 2.$

Note: Choosing a row or column with lots of zeros makes things easier!

Consider a matrix A & Let B denote what A becomes after each row op.

Recall: How do elementary row op's affect Matrices?

1. Multiply row by nonzero scalar K: $\det(B) = K\det(A)$.
2. Switch any 2 rows: $\det(B) = -\det(A)$.
3. Add a multiple of one row to an existing row: $\det(B) = \det(A)$.

Doing the same operations on A's columns yields the same results.

3\. Consider $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix}$. (a) Find $\det(A)$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{bmatrix} \xrightarrow{r_2 \leftarrow r_2 - 2r_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3 & 5 & 7 \end{bmatrix} = B.$$

$$\det(B) = -0 * \begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix} + 0 * \begin{vmatrix} 1 & 3 \\ 3 & 7 \end{vmatrix} - 0 * \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 0.$$

By (3) above, $\det(B) = \det(A) \rightarrow \det(A) = 0$.

<u>Note</u>: Any time we have a row or column of zeros, the determinant must always be 0.

(b) Find $\det\begin{pmatrix} t & 2t & 3t \\ 2 & 4 & 6 \\ 3 & 5 & 7 \end{pmatrix}$ for $t \in \mathbb{R}$.

Well, this matrix is just A followed by the row operation $r_1 \leftarrow r_1 * t$. So, by (1), $\det = t * \det(A)$ $= t*0 = 0$.

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4. Suppose $\det(A) = 3$, $\det(B) = 9$, $\det(C) = 2$.

What is $\det(X)$, if $BX = 6C^TA$.

Recall: We know the following properties concerning determinants:

(a) $\det(A) = \det(A^T)$
(b) $\det(AB) = \det(A)\det(B)$
(c) $\det(KA) = K^n \det(A)$, where $K \in \mathbb{R}$, & A is $n \times n$ matrix.
(d) $\det(A) \neq 0 \Leftrightarrow$ A invertible.
(e) $\det(A^{-1}) = \frac{1}{\det(A)}$.

$$BX = 6C^TA$$

* B^{-1} on left $\quad B^{-1}BX = 6B^{-1}C^TA \quad \rightsquigarrow B^{-1}$ exists, since $\det(B) = 9 \neq 0$ (by (d))

$\rightarrow X = 6B^{-1}C^TA$

$\rightarrow \det(X) = \det(6B^{-1}C^TA)$

$\rightarrow \det(X) = 6^n \det(B^{-1}C^TA) \rightsquigarrow$ by (c)

$= 6^n \det(B^{-1})\det(C^T)\det(A) \rightsquigarrow$ by (b)

$= 6^n * \frac{1}{\det(B)} * \det(C) * \det(A) \rightsquigarrow$ by (e) + (a)

$= 6^n * \frac{1}{9} * 2 * 3 = \frac{6^n * 2}{3}$. So, $\det(X) = \frac{2*6^n}{3}$.

5. Consider $A = \begin{bmatrix} 1 & x & 2 \\ 3 & 1 & -1 \\ -1 & 2 & 2 \end{bmatrix}$. (a) When is A singular?

<u>Recall</u>: A matrix A is called <u>singular</u> if A is <u>not</u> invertible. Also, we know that A invertible $\Leftrightarrow \det(A) \neq 0$, so A singular $\Leftrightarrow \det(A) = 0$.

So, we're looking for the values of x s.t. $\det(A) = 0$.

$\det(A) = 1 * \begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} - x * \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} + 2 * \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}$

$= 2 + +2 - x(6-1) + 2(6 + +1)$

$= 4 - 5x + 14 = -5x + 18.$

So $\det(A) = 0 \rightarrow -5x + 18 = 0 \rightarrow 5x = 18 \rightarrow x = \frac{18}{5}$.

So, x is singular when $x = \frac{18}{5}$.

(b) When is A invertible?

- A is invertible when $x \neq \frac{18}{5}$.

(b/c $x \neq \frac{18}{5} \rightarrow \det(A) \neq 0 \rightarrow$ A invertible).

6. Consider $A = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -3 & 1 \\ -2 & -1 & -2 \end{pmatrix}$. Find A^{-1} using the adjoint method.

<u>Recall</u>: If A is invertible, then $A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$,

where $\mathrm{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T$.

cofactors

$$\mathrm{adj}(A) = \begin{bmatrix} \begin{vmatrix} -3 & 1 \\ -1 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 1 \\ -2 & -2 \end{vmatrix} & \begin{vmatrix} -1 & -3 \\ -2 & -1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ -2 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ -2 & -1 \end{vmatrix} \\ \begin{vmatrix} 2 & 1 \\ -3 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ -1 & -3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} (6+1) & -(2+2) & (1-6) \\ -(-4+1) & (0+2) & -(0+4) \\ (2+3) & -(0+1) & (0+2) \end{bmatrix}^T = \begin{bmatrix} 7 & -4 & -5 \\ 3 & 2 & -4 \\ 5 & -1 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & 3 & 5 \\ -4 & 2 & -1 \\ -5 & -4 & 2 \end{bmatrix}.$$

Notice $A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)$

$\Rightarrow AA^{-1} = \frac{1}{\det(A)}\,A\,\mathrm{adj}(A)$

$\Rightarrow I\det(A) = A\,\mathrm{adj}(A)$.

$$\det(A) * I = A * \operatorname{adj}(A) = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -3 & 1 \\ -2 & -1 & -2 \end{bmatrix} \begin{bmatrix} 7 & 3 & 5 \\ -4 & 2 & -1 \\ -5 & -4 & 2 \end{bmatrix}$$

(good way to make sure you're right is @ this step... know you should get all 0's & same # on diagonal.)

$$= \begin{bmatrix} -13 & 0 & 0 \\ 0 & -13 & 0 \\ 0 & 0 & -13 \end{bmatrix} = -13 * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -13 * I$$

$\det(A) = -13.$

So, $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{-13} \begin{bmatrix} 7 & 3 & 5 \\ -4 & 2 & -1 \\ -5 & -4 & 2 \end{bmatrix}.$

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7. Solve the following linear system using Cramer's Rule: $\begin{cases} 3x + 2y = 1 \\ 5x + 4y = -1. \end{cases}$

__Recall:__ __Cramer's Rule:__ If $Ax = b$ is a system of n linear equations in n unknowns s.t. $\det A \neq 0$, then $Ax = b$ has a unique solution. This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, x_n = \frac{\det(A_n)}{\det(A)},$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix b.

Here we have $\underbrace{\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{b}$.

We have 2 equations with 2 unknowns, & $\det A = 3 \cdot 4 - 2 \cdot 5 = 2 \neq 0$, so we can use Cramer's Rule.

$A_1 = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. $\quad A_2 = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$.

$\det A_1 = 1 \cdot 4 - 2 \cdot (-1) = 6$.
$\det A_2 = 3 \cdot (-1) - 1 \cdot 5 = -8$.

So, $\quad x_1 = \dfrac{\det(A_1)}{\det(A)} = \dfrac{6}{2} = 3$.

$x_2 = \dfrac{\det(A_2)}{\det(A)} = \dfrac{-8}{2} = -4$.

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

<u>Check</u>

$\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. ✓

Tuesday Feb. 4th Tutorial: Suggested Problems

1 **Assignment #2: Problem #4:** Find the matrix A (in terms of B, C, & D), if $(4DA^{-1}B)^{-1} = C$.

$(4DA^{-1}B)^{-1} = C$

$\frac{1}{4}B^{-1}(A^{-1})^{-1}D^{-1} = C$

$\frac{1}{4} \cdot B^{-1}AD^{-1} = C$

$B^{-1}AD^{-1} = 4C$

$BB^{-1}AD^{-1} = B4C$

$IAD^{-1} = 4BC$

$AD^{-1} = 4BC$

$AD^{-1}D = 4BCD$

$AI = 4BCD$

$\boxed{A = 4BCD}$

$(AB)^{-1} = B^{-1}A^{-1}$. So, $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. Also, $K \in \mathbb{R} \Rightarrow K^{-1} = \frac{1}{K}$.

$(A^{-1})^{-1} = A$

* 4 on each side

* B on left

$AA^{-1} = I$

$IA = AI = A$

* D on left

$AA^{-1} = I$

$AI = IA = A$

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2 **Section 1.1: #15:** The curve $y = ax^2 + bx + c$ passes through the points (x_1, y_1), (x_2, y_2) & (x_3, y_3). Show a, b, & c are a solution to the linear system w/ augmented matrix:

15. $\begin{array}{cccc} u & v & w & \\ \end{array}$
$$\left[\begin{array}{ccc:c} x_1^2 & x_1 & 1 & y_1 \\ x_2^2 & x_2 & 1 & y_2 \\ x_3^2 & x_3 & 1 & y_3 \end{array}\right]$$

augmented matrix for linear system of eq'n's.

i.e.
$x_1^2 u + x_1 v + w = y_1$
$x_2^2 u + x_2 v + w = y_2$
$x_3^2 u + x_3 v + w = y_3.$

W.T.S. $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a solution.

(x_1, y_1) is on the curve $y = ax^2 + bx + c \iff y_1 = a x_1^2 + bx_1 + c.$

Similarly, (x_2, y_2) is on the curve $\iff y_2 = a x_2^2 + b x_2 + c$
& (x_3, y_3) " " $\iff y_3 = a x_3^2 + b x_3 + c.$

$\therefore$ We know $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a solution to this augmented matrix's corresponding system of eq'n's.

i.e.
$a x_1^2 + bx_1 + c(1) = y_1$ ✓
$a x_2^2 + bx_2 + c(1) = y_2$ ✓
$a x_3^2 + bx_2 + c(1) = y_3.$ ✓

(here we're treating the coefficients a, b, c like how we usually treat variables x, y, z).

i.e. $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a solution.