Math 1B03 - Tutorial #3

1. ⓐ Consider $A = \begin{bmatrix} 2 & -4 \\ -2 & 3 \end{bmatrix}$. Write A as a product of elementary matrices.

<u>Recall</u>: To do this we should:

① Reduce A to the identity I.
② Keep track of row operations.
③ Write each row operation as an elementary matrix.
④ Express the row reduction as matrix multiplication.
⑤ Solve for A.

$\begin{bmatrix} 2 & -4 \\ -2 & 3 \end{bmatrix} r_2 \leftarrow r_2 + r_1 \quad \begin{bmatrix} 2 & -4 \\ 0 & -1 \end{bmatrix} r_2 \leftarrow r_2 * -1$

$\begin{bmatrix} 2 & -4 \\ 0 & 1 \end{bmatrix} r_1 \leftarrow r_1 * \frac{1}{2} \quad \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} r_1 \leftarrow r_1 + 2r_2 \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$r_2 \leftarrow r_2 + r_1$ corresponds to $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$r_2 \leftarrow r_2 * -1 \longleftrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$r_1 \leftarrow r_1 * \frac{1}{2} \longleftrightarrow \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

$r_1 \leftarrow r_1 + 2r_2 \longleftrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

So, $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) Find A^{-1} without using the formula $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

We just showed that:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}} A = I$$

b/c $BA = I \rightarrow B = A^{-1}$.

$$\rightarrow A^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & -2 \\ -1 & -1 \end{bmatrix}.$$

Check

$$AA^{-1} = \begin{bmatrix} 2 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -\frac{3}{2} & -2 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Note: We went through #1 to demonstrate why the inverse algorithm works.

Inverse Algorithm: To find the inverse of an invertible matrix A:

1. Find a sequence of elementary row op's that reduce A to I_n.
2. Perform those same row op.'s on I_n to obtain A^{-1}!

These row op.'s can be written as elementary matrices

$$E_K E_{K-1} \cdots E_2 E_1 A = I \rightarrow A^{-1} = E_K E_{K-1} \cdots E_2 E_1 I.$$

So, to do this quickly, we perform the row op.'s represented by $E_K \cdots E_1$ simultaneously to A & I_n by right side becomes A^{-1} after row op's.

adjoining A with I_n: $[A \vdots I_n]$.

use row op's & reduce A to I_n.

$[A \vdots I_n]$

$\downarrow$

$[I_n \vdots A^{-1}]$.

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2. Consider $A = \begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 5 \\ 3 & 2 & 3 \end{pmatrix}$.

Recall that we found that $A^{-1} = \begin{bmatrix} -11 & 1 & 2 \\ 3 & 0 & 1 \\ 9 & -1 & -1 \end{bmatrix}$ by using row operations. (see week 2 notes).

ⓐ Does $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ have a unique solution?

<u>Recall</u>: We know several equivalent statements where A is a $n \times n$ matrix:

ⓐ A is invertible.
ⓑ $Ax = 0$ has only the trivial solution.
ⓒ The reduced row echelon form of A is I_n.
ⓓ A is expressible as the product of elementary matrices.
ⓔ $Ax = b$ is consistent for every $n \times 1$ matrix b.
ⓕ $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.

Well, we know that A is invertible, & $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a 3×1 matrix, so ⓐ → ⓕ, i.e., $Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has exactly one solution since A is invertible.

(b) Does $Ax = 0$ have non-trivial solutions?

- Yes!

We know (b) → (a), so ~(a) → ~(b).

i.e., Since A is <u>not</u> invertible, $Ax = 0$ does <u>not</u> only have the trivial solution → $Ax = 0$ has non trivial solutions.

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4. (a) Consider $B = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 7 & 2 \\ 5 & a & 9 \end{bmatrix}$. What must "a" be in order for B to be symmetric?

<u>Recall</u>: A square matrix B is called <u>symmetric</u> if $B = B^T$.

$B^T = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 7 & a \\ 5 & 2 & 9 \end{bmatrix}$, so we must have $a = 2$ in order for B to be symmetric.

(b) Let A, B, C, & D be symmetric. Is $A+B+C+D$ symmetric?

$$(A+B+C+D)^T = \underbrace{A^T + B^T + C^T + D^T}_{\text{transpose rules}} = \underbrace{A + B + C + D}_{A,B,C,D \text{ symmetric}}$$

→ $A + B + C + D$ symmetric.

(c) Consider $C = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Find BC.

$$BC = \begin{bmatrix} 2 & 12 & 20 \\ 8 & 21 & 8 \\ 10 & 3a & 36 \end{bmatrix} = \begin{bmatrix} 2*1 & 3*4 & 4*5 \\ 2*4 & 3*7 & 4*2 \\ 2*5 & 3*a & 4*9 \end{bmatrix}$$

notice that multiplying a matrix by a diagonal matrix is easy!

(b) Find X.

$$Ax = \begin{bmatrix}1\\2\\3\end{bmatrix} \rightarrow X = A^{-1}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-11 & 1 & 2\\3 & 0 & -1\\9 & -1 & -1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$= \begin{bmatrix}-3\\0\\4\end{bmatrix}.$$

Check $\begin{bmatrix}1 & 1 & 1\\6 & 7 & 5\\3 & 2 & 3\end{bmatrix}\begin{bmatrix}-3\\0\\4\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ ✓

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3. Consider $A = \begin{bmatrix}1 & 2 & 3\\2 & 4 & 6\\1 & 7 & 1\end{bmatrix}$.

(a) Is A invertible?

Let's try to find A^{-1}:

$$\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\2 & 4 & 6 & 0 & 1 & 0\\1 & 7 & 1 & 0 & 0 & 1\end{array}\right] r_2 \leftarrow r_2 - 2r_1$$

$$\left[\begin{array}{ccc|ccc}1 & 2 & 3 & 1 & 0 & 0\\0 & 0 & 0 & -2 & 1 & 0\\1 & 7 & 1 & 0 & 0 & 1\end{array}\right]$$

No!

We have a row of zeros! By the equivalence statements we know A invertible $\Leftrightarrow$ the reduced row echelon form of A is I_n. I_n does not have a row of zeros, so r.r.e.f. of A $\neq I_n$ $\rightarrow$ A is not invertible. i.e., $\sim$(c) $\rightarrow$ $\sim$(a).

here $\sim$ denotes "not".