Math 1B03 - Tutorial #2

1. Let $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$.

(a) Is A invertible?

- $\det(A) = 5 - 6 = -1 \neq 0 \rightarrow A$ is invertible.

(b) Find A^{-1}!

- $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 5 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$.

Check $\begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

(c) Is B invertible?

- $\det(B) = 16 - 15 = 1 \neq 0 \rightarrow B$ is invertible.

(d) Find B^{-1}!

- $B^{-1} = \frac{1}{1} \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix}$.

Check $\begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ✓

(e) Find $(AB)^{-1}$!

↳ We know $(AB)^{-1} = B^{-1}A^{-1}$ (p.g. 45)

$\rightarrow (AB)^{-1} = \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -50 & 29 \\ 19 & -11 \end{pmatrix}$.

2. Let $A = \begin{pmatrix} 4 & x \\ x & 1 \end{pmatrix}$. For which values of x is A singular.

- We know A is singular $\Leftrightarrow$ $\det(A) = 0$.

$\det(A) = 4 - x^2 = 0 \Leftrightarrow x^2 = 4 \Leftrightarrow x = 2$ or $x = -2$.

So, A will be singular if $x = 2$ or $x = -2$.

3. Solve for x: $A(x + B) = CA$. (where A is invertible).

$$A(X+B) = CA$$
$$A^{-1}A(X+B) = A^{-1}CA \qquad * \ A^{-1} \text{ on left}$$
$$I(X+B) = A^{-1}CA \qquad A^{-1}A = I$$
$$X+B = A^{-1}CA \qquad I * J = J$$
$$X+B-B = A^{-1}CA - B \qquad \text{Minus } B \text{ on each side}$$
$$X = A^{-1}CA - B.$$

4. Solve for X: $(2E+F)^T = G^{-1}X^T + F^T$.

(see pg.'s 47 & 48 for transpose & inverse properties).

$$(2E+F)^T = G^{-1}X^T + F^T$$
$$(2E)^T + F^T = G^{-1}X^T + F^T \qquad (A+B)^T = A^T + B^T$$
$$2E^T + F^T = G^{-1}X^T + F^T \qquad (4A)^T = 4A^T$$
$$2E^T + F^T - F^T = G^{-1}X^T + F^T - F^T \qquad \text{minus } F^T \text{ from both sides}$$
$$2E^T = G^{-1}X^T$$
$$2GE^T = GG^{-1}X^T \qquad * \ G \text{ on left}$$
$$2GE^T = X^T \qquad GG^{-1} = I$$
$$(2GE^T)^T = (X^T)^T \qquad \text{take transpose of both sides}$$
$$2(GE^T)^T = X \qquad (A^T)^T = A \ \& \ (4A)^T = 4A^T$$
$$2(E^T)^T G^T = X \qquad (AB)^T = B^T A^T$$
$$2EG^T = X$$

5. Find the inverse of $A = \begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 5 \\ 3 & 2 & 3 \end{pmatrix}$ using row operations.

$$\left(\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 6 & 7 & 5 & 0 & 1 & 0 \\ 3 & 2 & 3 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} r_2 \leftarrow r_2 - 6r_1 \\ r_3 \leftarrow r_3 - 3r_1 \end{array} \qquad \left(\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -6 & 1 & 0 \\ 0 & -1 & 0 & -3 & 0 & 1 \end{array}\right) \begin{array}{l} r_1 \leftarrow r_1 - r_2 \\ r_3 \leftarrow r_3 + r_1 \end{array}$$

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 2 & 7 & -1 & 0 \\ 0 & 1 & -1 & -6 & 1 & 0 \\ 0 & 0 & -1 & -9 & 1 & 1 \end{array}\right) r_3 \leftarrow r_3 * -1 \qquad \left(\begin{array}{ccc:ccc} 1 & 0 & 2 & 7 & -1 & 0 \\ 0 & 1 & -1 & -6 & 1 & 0 \\ 0 & 0 & 1 & 9 & -1 & -1 \end{array}\right) \begin{array}{l} r_2 \leftarrow r_2 + r_3 \\ r_1 \leftarrow r_1 - 2r_3 \end{array}$$

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 0 & -11 & 1 & 2 \\ 0 & 1 & 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 9 & -1 & -1 \end{array}\right).$$

So, $A^{-1} = \begin{pmatrix} -11 & 1 & 2 \\ 3 & 0 & -1 \\ 9 & -1 & -1 \end{pmatrix}$.

<u>Check</u>

$$\begin{pmatrix} 1 & 1 & 1 \\ 6 & 7 & 5 \\ 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} -11 & 1 & 2 \\ 3 & 0 & -1 \\ 9 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

6. (a) Solve for W: $2EWF^2 = (E^TF)^2$, (where E & F are invertible & are the same size).

$$2EWF^2 = E^TFE^TF \qquad (AB)^2 = ABAB$$

$$EWF^2 = \tfrac{1}{2}E^TFE^TF \qquad * \tfrac{1}{2} \text{ on both sides}$$

$$E^{-1}EWF^2 = \tfrac{1}{2}E^{-1}E^TFE^TF \qquad * E^{-1} \text{ on left (supposing E is invertible)}$$

$$WF^2 = \tfrac{1}{2}E^{-1}E^TFE^TF \qquad E^{-1}E = I$$

$$WFFF^{-1} = \tfrac{1}{2}E^{-1}E^TFE^TFF^{-1} \qquad * F^{-1} \text{ on right}$$

$$WF = \tfrac{1}{2}E^{-1}E^TFE^T \qquad FF^{-1} = I$$

$$WFF^{-1} = \tfrac{1}{2}E^{-1}E^TFE^TF^{-1} \qquad * F^{-1} \text{ on right}$$

$$W = \tfrac{1}{2}E^{-1}E^TFE^TF^{-1}. \qquad FF^{-1} = I$$

(b) What sizes must F & W be in order for W to have a unique solution if E is 3xn.

- Well, in order for E to be invertible, we would need E to be square → E is 3x3 → F is 3x3 in order for E^TFE^T to be defined → W is 3x3, since if E^{-1} exists, then $W = \tfrac{1}{2}\,{}_{3}E^{-1}E^TFE^TF^{-1}{}_{3}$.

Note that if E is <u>not</u> invertible, then W may not be unique. Indeed, we know that there is no cancellation law for matrices, so $AB = AC$ doesn't necessarly mean that $B = C$. Let $Z = \tfrac{1}{2}E^TFE^TF^{-1}$. So, if $EW_1 = EW = Z$ we may have $W_1 \neq W$.

1.2: #25: Determine the values of a for which the system has (a) no solutions, (b) exactly one solution, (c) infinitely many solutions.

$$x + 2y - 3z = 4$$
$$3x - y + 5z = 2$$
$$4x + y + (a^2-14)z = a+2.$$

$$\left[\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right] \begin{array}{l} r_2 \leftarrow r_2 - 3r_1 \\ r_3 \leftarrow r_3 - 4r_1 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right] r_3 \leftarrow r_3 - r_2$$

$$\left[\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

$a^2 - 16 = 0 \Leftrightarrow a^2 = 16 \Leftrightarrow a = 4$ or $a = -4$.
$a - 4 = 0 \Leftrightarrow a = 4$.

If $a = 4$ my system becomes: $\left[\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

free parameters = # unknowns − # leading 1's
$= 3 - 2 = 1.$

$\therefore$ If $a=4$ I'll get a free parameter $\Rightarrow$ infinitely many solutions.

If $a = -4$ my system becomes: $\left[\begin{array}{ccc:c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & 0 & -8 \end{array}\right]$.

$0x + 0y + 0z = -8$ is a contradiction
$\Rightarrow$ no solution ∎

If $a \neq 4$ & $a \neq -4$, then I'll have exactly one solution, because row op.'s will lead to 3 leading 1's.

<u>Partitioned Matrices</u> (see pg. 30 of textbook).

Suppose you multiply two matrices AB.

<u>Notice:</u> j^{th} column of AB = $A * [j^{th}$ column of $B]$

<u>i.e.,</u> Suppose B is $m \times p$.

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & & & \vdots \\ \vdots & & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mp} \end{bmatrix} = [c_1 \ c_2 \ \cdots \ c_p].$$

$c_1 \quad c_2 \quad \cdots \quad c_p$

notation

Then $AB = A [c_1 \ c_2 \ \cdots \ c_p] = [Ac_1 \ Ac_2 \ \cdots \ Ac_p].$

(to see this, write out A & B like how we wrote B as $B = \begin{bmatrix} b_{11} \cdots b_{1p} \\ \vdots \\ b_{m1} \cdots b_{mp} \end{bmatrix}$ above, & do the multiplication ☺)

Note: Just b/c we get a row of zeros does <u>not</u> necessarily mean we'll have infinitely many solutions.

<u>e.g.</u> ⇒ Consider the system
$$\begin{cases} x = 1 \\ y = 2 \\ z = 3 \\ w = 4 \\ x + y + z + w = 10. \end{cases}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 1 & 1 & 1 & 1 & 10 \end{array}\right]$$

Do the row op.'s
$r_5 \leftarrow r_5 - r_1$
$r_5 \leftarrow r_5 - r_2$
$r_5 \leftarrow r_5 - r_3$
$r_5 \leftarrow r_5 - r_4$.

You'll get:
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

free parameters = # unknowns − # leading 1's
= 4 − 4 = 0 ⇒ exactly one solution ♡

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Does the Commutative Law for Multiplication hold for Matrices?

- What if A and B are both square (i.e. A and B are both $n \times n$ matrices)?
- Does $AB = BA$ for any possible A and B?
- Can you think of a counterexample?

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

not same!

Does the Commutative Law for Multiplication hold for Matrices?

- Is it ever possible to find an A and B such that $AB = BA$?

L Yes

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 12 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

Zero Divisors?

- For real numbers, we know that $ab = 0 \Rightarrow a = 0$ or $b = 0$.
- Is this true for matrices? (i.e. if we have two matrices A and B such that $AB = 0$, is it true that we must have $A = 0$ or $B = 0$?)

No!

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Notice $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ is not defined.

So we can also see that $AB = 0$ doesn't imply that $BA = 0$.

Cancellation Law?

- For real numbers, we know that $ab = ac \Rightarrow b = c$.
- Does this hold true in general for matrices? (i.e. $AB = AC \Rightarrow B = C$?

No.

$$AB = \begin{pmatrix} 2 & -6 \\ 6 & -18 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$AC = \begin{pmatrix} 2 & -6 \\ 6 & -18 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{but} \quad \begin{pmatrix} 3 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 6 \\ 2 \end{pmatrix}.$$

Multiplicative Inverse

- In $\mathbb{R}$ we know that for every a such that $a \neq 0$ there exists a^{-1} such that $aa^{-1} = a^{-1}a = 1$. **e.g** $2 \times \frac{1}{2} = 1 = \frac{1}{2} \times 2$.
- If A is a square $(n \times n)$ matrix such that $\exists$ a B such that $AB = I_{n \times n} = BA$, then A is said to be **invertible**, (a.k.a **nonsingular**), and B is called the inverse of A, $(B = A^{-1})$.
- If A is a 2×2 matrix, then

$$A^{-1} = \frac{1}{ad-bc} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

b/c:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\frac{1}{ad-bc} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}\frac{1}{ad-bc} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$