Math 1803 - Tutorial #11

1. (see #5 from Tutorial #10 Notes).

2. Which of the following form a basis for $\mathbb{R}^2$?

Recall: • A set $S = \{v_1, \dots, v_n\}$ of vectors, where $v_1, \dots, v_n \in V$ is called a **basis** for V if:
1. S is linearly independent.
2. S spans V.

• $S = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ is called the **standard basis** for $\mathbb{R}^2$.

Theorem 4.5.2 Pg. 209

• If $\{v_1, \dots, v_n\}$ is a basis for V, then:
(a) If a set S of vectors from V has $> n$ vectors $\Rightarrow S$ linearly dependent.
(b) If S has $< n$ vectors $\Rightarrow S$ does not span V.

(a) $S = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix} \right\}$.

Using the Theorem, we know $\mathbb{R}^2$ has a basis of 2 vectors $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$, but S has 3 vectors. $3 > 2 \Rightarrow S$ is not linearly independent $\Rightarrow S$ is not a basis for $\mathbb{R}^2$.

We could have also solved this directly:

S is a linearly independent set if $k_1 \begin{pmatrix}1\\0\end{pmatrix} + k_2 \begin{pmatrix}1\\1\end{pmatrix} + k_3 \begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ has only the trivial solution; i.e. $\begin{bmatrix}1 & 1 & 0\\0 & 1 & 2\end{bmatrix} \begin{bmatrix}k_1\\k_2\\k_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ has only the trivial solution.

$$\left[\begin{array}{ccc|c}1 & 1 & 0 & 0\\0 & 1 & 2 & 0\end{array}\right] \qquad \begin{array}{l} k_1 = -k_2 \\ 2k_3 = -k_2 \end{array} \qquad k_2 = t$$

← Parameter ⇒ infinitely many solutions ⇒ not lin. independent.

However, S does span $\mathbb{R}^2$:

In order for S to span $\mathbb{R}^2$, we need that given any $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ there exists $k_1, k_2, k_3 \in \mathbb{R}$ s.t.

$$k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_3 \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & a \\ 0 & 1 & 2 & b \end{array}\right] \qquad \begin{array}{l} k_1 = -k_2 + a \Rightarrow k_1 = -t + a \\ 2k_3 = -k_2 + b \Rightarrow k_3 = -\frac{1}{2}t + b/2 \\ k_2 = t \end{array}$$

So, this system has solution $\begin{pmatrix} a \\ 0 \\ b/2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1/2 \end{pmatrix} t$

$\Rightarrow S$ spans $\mathbb{R}^2$.

So, S spans $\mathbb{R}^2$, but S is <u>not</u> linearly independent
$\Rightarrow S$ is <u>not</u> a basis for $\mathbb{R}^2$.

(b) $T = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

T is linearly independent if $k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ only has the trivial solution.

T spans $\mathbb{R}^2$ if given any $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$ $\exists$ $k_1, k_2 \in \mathbb{R}$ s.t. $\begin{pmatrix} a \\ b \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Let's solve these simultaneously:

$$\left[\begin{array}{cc:cc} 1 & 1 & 0 & a \\ 1 & -1 & 0 & b \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 + r_2} \left[\begin{array}{cc:cc} 2 & 0 & 0 & a+b \\ 1 & -1 & 0 & b \end{array}\right]$$

$$\begin{array}{l} 2k_1 = 0 \Rightarrow k_1 = 0. \\ k_1 - k_2 = 0 \\ \Rightarrow 0 - k_2 = 0 \\ \Rightarrow k_2 = 0. \end{array} \quad \Bigg| \quad \begin{array}{l} 2k_1 = a + b \\ \Rightarrow k_1 = \frac{a}{2} + \frac{b}{2}. \\ k_1 - k_2 = b \\ \Rightarrow k_2 = \frac{a}{2} + \frac{b}{2} - b \\ \Rightarrow k_2 = \frac{a}{2} - b/2. \end{array}$$

So, $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ only has the trivial solution $\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow T$ linearly independent.

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ has the solution $\begin{bmatrix} \frac{a}{2} + \frac{b}{2} \\ \frac{a}{2} - \frac{b}{2} \end{bmatrix}$ for any $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \Rightarrow T$ spans $\mathbb{R}^2$.

$\therefore$ T is a basis for $\mathbb{R}^2$.

3. We know $T = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ forms a basis for $\mathbb{R}^2$. (by the previous exercise #2(b)).

(a) Find the coordinate vector of $v = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ relative to the basis T; i.e., Find $[v]_T$.

<u>Recall:</u> If $S = \{v_1, \dots, v_n\}$ is a basis for V, and $w = K_1 v_1 + \dots + K_n v_n$ for $K_1, \dots, K_n \in \mathbb{R}$, then $[w]_S = (K_1, \dots, K_n)$ is called the <u>coordinate vector of v relative to S</u>.

So, we want to find the $K_1, K_2 \in \mathbb{R}$ such that

$$K_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + K_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

$\left[\begin{array}{cc:c} 1 & 1 & 3 \\ 1 & -1 & 5 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 + r_2} \left[\begin{array}{cc:c} 2 & 0 & 8 \\ 1 & -1 & 5 \end{array}\right]$ $\quad 2K_1 = 8 \Rightarrow K_1 = 4$

$K_1 - K_2 = 5 \Rightarrow K_2 = 4 - 5 = -1$.

So $4\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \Rightarrow [v]_T = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

(b) Find the vector $w \in \mathbb{R}^2$ whose coordinate vector relative to T is $[w]_T = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

By def'n $[w]_T = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ means that

$w = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow w = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$.

4. Which of the following are a basis for P_2 (where P_2 is the vectorspace of all polynomials of degree ≤ 2; i.e. $P_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbb{R}\}$.

$W = \{x^2, x+1, x^2+x+1\}$
$X = \{x, 1, 0\}$
$T = \{x^2+x+1, x^2+x, x+1\}$
$Y = \{x^2, x, 1, 0\}$.

Y: We know $\{1, x, x^2\}$ is the standard basis for P_2 & this basis has 3 elements. Therefore, using Theorem 4.5.2 we can see that Y is a linearly dependent set since it has 4 elements, and $4 > 3$. So, Y not linearly independent $\Rightarrow$ Y not a basis for P_2.

X: We can see $c_1(0) + c_2(1) + c_3(x)$ will never equal x^2 for $c_1, c_2, c_3 \in \mathbb{R} \Rightarrow$ X does not span $P_2 \Rightarrow$ X is not a basis for P_2.

W: Now, looking at W we can see that $(x^2) + (x+1) = (x^2+x+1)$ $\Rightarrow$ W is not an independent set $\Rightarrow$ W is not a basis for P_2. (i.e. $(-1)x^2 + (-1)(x+1) + (1)(x^2+x+1) = 0$, & $(-1,-1,1)$ is not the trivial solution $\Rightarrow$ not linearly independent.)

$\underline{T}$: Now, looking at T, nothing obviously wrong is jumping out, so let's formally check whether T is linearly independent, & whether or not T spans P_2:

Linear Independence: Suppose $K_1(x^2+x+1) + K_2(x^2+x) + K_3(x+1) = 0$

$\Rightarrow (K_1+K_2)x^2 + (K_1+K_2+K_3)x + (K_1+K_3) = 0$

$\Rightarrow K_1+K_2 = 0$ & $K_1+K_2+K_3 = 0$ & $K_1+K_3 = 0$

Want to solve: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 - r_1} \left[\begin{array}{ccc:c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$

$K_3 = 0$
$K_1 + K_3 = 0 \Rightarrow K_1 = 0$
$K_1 + K_2 = 0 \Rightarrow K_2 = 0.$

So, $K_1(x^2+x+1) + K_2(x^2+x) + K_3(x+1) = 0$ only has the solution $(K_1, K_2, K_3) = (0,0,0) \Rightarrow$ T linearly independent.

Spanning: We want to know if for each $a+bx+cx^2 \in P_2$ $\exists\ K_1, K_2, K_3 \in \mathbb{R}$ s.t. $K_1(x^2+x+1) + K_2(x^2+x) + K_3(x+1) = a+bx+cx^2$?

i.e. $\Rightarrow (K_1+K_2)x^2 + (K_1+K_2+K_3)x + (K_1+K_3) = a+bx+cx^2$

$\Leftrightarrow K_1+K_2 = c,\ K_1+K_2+K_3 = b,\ K_1+K_3 = a$

$\left[\begin{array}{ccc:c} 1 & 1 & 0 & c \\ 1 & 1 & 1 & b \\ 1 & 0 & 1 & a \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 - r_1} \left[\begin{array}{ccc:c} 1 & 1 & 0 & c \\ 0 & 0 & 1 & b-c \\ 1 & 0 & 1 & a \end{array}\right]$

$K_3 = b-c$
$K_1 = a - K_3 = a-b+c$
$K_2 = c - K_1 = c-(a-b+c)$
$= -a+b.$

So, $\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{bmatrix} a - b + c \\ -a + b \\ b - c \end{bmatrix} \Rightarrow$ For each $a + bx + cx^2 \in P_2$

we have a solution $\Rightarrow$ T spans P_2.

$\therefore$ T is independent & spans $P_2 \Rightarrow T$ is a basis for P_2.

4.2: #9: Which of the following are linear combinations of

$$A = \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} ?$$

(a) $\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$, (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 6 & 0 \\ 3 & 8 \end{bmatrix}$, (d) $\begin{bmatrix} -1 & 5 \\ 7 & 1 \end{bmatrix}$.

(a) (a) is a linear combination of A, B, & C if $\exists$ scalars $K_1, K_2, K_3 \in \mathbb{R}$ s.t. $K_1 A + K_2 B + K_3 C =$ (a).

i.e.) $K_1 \begin{bmatrix} 4 & 0 \\ -2 & -2 \end{bmatrix} + K_2 \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} + K_3 \begin{bmatrix} 0 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix}$

$$\Leftrightarrow \begin{aligned} 4K_1 + K_2 + 0K_3 &= 6 \\ 0K_1 - K_2 + 2K_3 &= -8 \\ -2K_1 + 2K_2 + K_3 &= -1 \\ -2K_1 + 3K_2 + 4K_3 &= -8 \end{aligned} \quad \Leftrightarrow \begin{bmatrix} 4 & 1 & 0 \\ 0 & -1 & 2 \\ -2 & 2 & 1 \\ -2 & 3 & 4 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -1 \\ -8 \end{bmatrix}.$$

So, we need to solve:

$$\left[\begin{array}{ccc:c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{array}\right] \begin{array}{l} \\ \\ r_3 \leftarrow r_3 + 2r_2 \\ r_4 \leftarrow r_4 + 3r_2 \end{array}$$

$$\left[\begin{array}{ccc:c} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 0 & 5 & -17 \\ -2 & 0 & 10 & -32 \end{array}\right] \begin{array}{l} r_1 \leftarrow r_1 + r_2 \\ r_4 \leftarrow r_4 - r_3 \end{array}$$

$$\left[\begin{array}{ccc:c} 4 & 0 & 2 & -2 \\ 0 & -1 & 2 & -8 \\ -2 & 0 & 5 & -17 \\ 0 & 0 & 5 & -15 \end{array}\right] \begin{array}{l} \\ \\ r_1 \leftarrow r_1 \cdot \tfrac{1}{2} \\ r_3 \leftarrow r_3 - r_4 \end{array}$$

$$\left[\begin{array}{ccc:c} 2 & 0 & 1 & -1 \\ 0 & -1 & 2 & -8 \\ -2 & 0 & 0 & -2 \\ 0 & 0 & 5 & -15 \end{array}\right] r_3 \leftarrow r_3 + r_1 \qquad \left[\begin{array}{ccc:c} 2 & 0 & 1 & -1 \\ 0 & -1 & 2 & -8 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 5 & -15 \end{array}\right] r_4 \leftarrow r_4 - 5r_3$$

$$\left[\begin{array}{ccc:c} 2 & 0 & 1 & -1 \\ 0 & -1 & 2 & -8 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$2K_1 + K_3 = -1 \Leftrightarrow 2K_1 - 3 = -1 \Leftrightarrow 2K_1 = 2 \Leftrightarrow K_1 = 1$

$-K_2 + 2K_3 = -8 \Leftrightarrow -K_2 - 6 = -8 \Leftrightarrow -K_2 = -2 \Leftrightarrow K_2 = 2$

$K_3 = -3.$

So, $\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ solves this system.

i.e. $A + 2B - 3C = \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} \Rightarrow$ (a) is a linear combination of $A, B, \text{ \& } C$.

(b) Obviously $0A + 0B + 0C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow$ (b) is a linear combination of $A, B, \text{ \& } C$.

(c) & (d) Set up a system like in (a) & see whether or not a solution for K_1, K_2, K_3 exists.