Math 1B03 - Tutorial #10

1. Let $V = \mathbb{R}^2$ & define addition and scalar multiplication as follows: If $u = (x_1, y_1)$, $v = (x_2, y_2)$, then

$$u + v := \begin{pmatrix} x_1 - 2x_2 + 1 \\ 2y_1 + 3y_2 - 4 \end{pmatrix}$$

$$\alpha * u = \begin{pmatrix} \frac{1}{\alpha} x_1 \\ y_1 \alpha^2 \end{pmatrix}.$$

Is V a vector space with these stated operations? Specify which axioms hold, and which fail.

Note: Our scalars don't have to be in $\mathbb{R}$, but for simplicity, I'll use $\mathbb{R}$ here.

Recall: A vectorspace is a set V together with a binary operation "+" and a rule for scalar multiplication satisfying 10 axioms. i.e. If the axioms hold $\forall$ vectors $v, u, w \in V$ & $\forall$ scalars $\alpha, \beta \in \mathbb{R}$, then V is a vector space.

Vector Space Axioms

"+" = Additive
"α" Scalar

1. "+" Closure: $v, w \in V \rightarrow v + w \in V$.
2. "+" Commutativity: $v, w \in V \rightarrow v + w = w + v$.
3. "+" Associativity: $u, v, w \in V \rightarrow (u+v) + w = u + (v+w)$
4. "+" Identity: $\exists$ a vector $\bar{0} \in V$ s.t. $v + \bar{0} = v \quad \forall v \in V$.
5. "+" Inverse: For each $v \in V$ $\exists$ $(-v) \in V$ s.t. $v + (-v) = \bar{0}$.
6. "α" Closure: $v \in V \rightarrow \alpha v \in V \quad \forall \alpha \in \mathbb{R}$.
7. "α" Distributivity: $\alpha(v + w) = \alpha v + \alpha w \quad \forall v, w \in V, \alpha \in \mathbb{R}$.
8. Vector Distributivity: $(\alpha + \beta) v = \alpha v + \beta v \quad \forall v \in V, \alpha, \beta \in \mathbb{R}$.
9. "α" Associativity: $\alpha(\beta v) = (\alpha\beta) v \quad \forall v \in V, \alpha, \beta \in \mathbb{R}$.
10. "α" Identity: $1 * v = v \quad \forall v \in V, 1 \in \mathbb{R}$.

$$\boxed{u+v := \begin{pmatrix} x_1 - 2x_2 + 1 \\ 2y_1 + 3y_2 - 4 \end{pmatrix}}$$

"+" Closure ① Let $u, v \in V$. Then $u+v = \underbrace{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}}_{u} + \underbrace{\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}}_{v}$

$= \begin{pmatrix} x_1 - 2x_2 + 1 \\ 2y_1 + 3y_2 - 4 \end{pmatrix} \in \mathbb{R}^2 = V.$ ✓

"+" Comm. ② $v+u = \begin{pmatrix} x_2 - 2x_1 + 1 \\ 2y_2 + 3y_1 - 4 \end{pmatrix}.$

$u+v = \begin{pmatrix} x_1 - 2x_2 + 1 \\ 2y_1 + 3y_2 - 4 \end{pmatrix}.$

If $u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then

$u+v = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$ Not same!

$v+u = \begin{pmatrix} 1+1 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$. $x+y \neq y+x$.

So, ② Fails. $\therefore$ V is not a vector space.

"+" Assoc. ③ By inspection we may guess that this will fail. Let's check. Let $u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then,

$(u+v)+z = \begin{pmatrix} -1 \\ -4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+1 \\ -8+3-4 \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \end{pmatrix}$ Not same!

$u+(v+z) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1+1 \\ 3-4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0-4+1 \\ 0-3-4 \end{pmatrix} = \begin{pmatrix} -3 \\ -7 \end{pmatrix}$

So, $\exists\, u, v, z \in V$ s.t. ③ fails $\rightarrow$ ③ is not true.

Note: Remember that if you want to prove something is not true, then you just need to find a counter example. But, to prove something is true you must prove it in general.

:) true means true $\forall v$. Not true $\exists\, v \in V$ s.t. v fails.

"+" Identity ④ Let $u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$. Does $\exists!\, \bar{0} \in V$ s.t. $u + \bar{0} = u$?
Let $\bar{0} = \begin{pmatrix} a \\ b \end{pmatrix}$.

Then $u + \bar{0} = \begin{pmatrix} x_1 - 2a + 1 \\ 2y_1 + 3b - 4 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}.$

$\rightarrow x_1 - 2a + 1 = x_1 \rightarrow 2a = 1 \rightarrow a = \frac{1}{2}.$

$\rightarrow 2y_1 + 3b - 4 = y_1 \rightarrow 3b = -y_1 + 4 \rightarrow b = -\frac{1}{3}y_1 + \frac{4}{3}.$

Note: Indeed, $\bar{0}$ must be unique, b/c if $\exists\ \tilde{0} \in V$ s.t. $v+\tilde{0}=v$ & $v+\bar{0}=v$ → $v+\bar{0}=v+\tilde{0}$ → $(-v)+v+\bar{0}=(-v)+v+\tilde{0}$ → $\bar{0}+\bar{0}=\tilde{0}+\tilde{0}$ → $\bar{0}=\tilde{0}$.

So, $\bar{0}=\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{3}y_1+\frac{4}{3} \end{pmatrix}$. But since we have a variable y_1, this means that $\bar{0}$ is not unique.

∴ 4 is False.

5 "+" Inverse

Well, $\bar{0}$ does not exist, so 5 cannot be true. i.e., there is no $(-v)$ s.t. $(-v)+v=\bar{0}$. ∴ 5 is False.

"α" Closure

6 $\alpha u = \alpha\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha} x_1 \\ x_2 \alpha^2 \end{pmatrix}$.

We can see that if $\alpha=0$, then we divide by 0. So $\exists\ \alpha \in \mathbb{R}$ s.t. $\alpha \cancel{u} \notin \mathbb{R}^2$.

∴ 6 Fails.

"α" Distributivity

7 $\alpha(u+v) = \alpha\begin{pmatrix} x_1-2x_2+1 \\ 2y_1+3y_2-4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\alpha}(x_1-2x_2+1) \\ (2y_1+3y_2-4)\alpha^2 \end{pmatrix}$.

$\alpha u + \alpha v = \begin{pmatrix} \frac{1}{\alpha}x_1 \\ y_1\alpha^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{\alpha}x_2 \\ y_2\alpha^2 \end{pmatrix}$.

(of course, it's a little silly to check 7 - 10, since $\frac{1}{\alpha}$ doesn't exist, but if we ignore this problem, it's good practice.)

Now, I'm going to first make a guess & see if I can find a u, v, α s.t. these are not equal. (As in this case, it's usually easier to show that something is not true, than true.)

Let $u=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $v=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\alpha=2$. Then, Not same!

$\alpha(u+v)=\begin{pmatrix} \frac{1}{2} \\ -16 \end{pmatrix}$. $\alpha u+\alpha v=\begin{pmatrix} 0 \\ 0 \end{pmatrix}+\begin{pmatrix} 0 \\ 0 \end{pmatrix}=\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

∴ 7 is False.

$$u+v = \begin{pmatrix} x_1 - 2x_2\,1 \\ 2y_1 + 3y_2 - 4 \end{pmatrix}$$

Let $v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\alpha = 1$, $\beta = 2$:

Vector Distributivity ⑧

$$(\alpha + \beta)u = \begin{pmatrix} \frac{x_1}{\alpha+\beta} \\ y_1(\alpha+\beta)^2 \end{pmatrix}$$

$$\alpha u + \beta u = \begin{pmatrix} \frac{x_1}{\alpha} \\ y_1\alpha^2 \end{pmatrix} + \begin{pmatrix} \frac{x_1}{\beta} \\ y_1\beta^2 \end{pmatrix}$$

$$(\alpha+\beta)u = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Not same!

$$\alpha u + \beta u = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

$\therefore$ 8 is False.

"α" Assoc. ⑨

$$\alpha(\beta u) = \alpha \begin{pmatrix} \frac{x_1}{\beta} \\ y_1\beta^2 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{\alpha\beta} \\ y_1\alpha^2\beta^2 \end{pmatrix}$$

same!

$$\alpha\beta(u) = \begin{pmatrix} \frac{x_1}{\alpha\beta} \\ y_1(\alpha\beta)^2 \end{pmatrix} = \begin{pmatrix} \frac{x_1}{\alpha\beta} \\ y_1\alpha^2\beta^2 \end{pmatrix}$$

$\therefore$ 9 is true! (or, I should say it would be true if 6 didn't Fail).

"α" Identity ⑩

$$1 * u = \begin{pmatrix} \frac{x_1}{1} \\ y_1(1)^2 \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = u \checkmark$$

$\therefore$ 10 is true!

2. If $V = \mathbb{R}^2$ is a set with addition & scalar multiplication defined as $u+v = (u_1+v_1+1, u_2+v_2+1)$, $\alpha u = (\alpha u_1, \alpha u_2)$, then what must $\bar{0}$ be?

Let $\bar{0} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then $v + \bar{0} = v$

$$\Leftrightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} u_1 + a + 1 \\ u_2 + b + 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Leftrightarrow \begin{matrix} a = -1 \\ b = -1. \end{matrix}$$

So, $\bar{0} = (-1, -1)$.

<u>Note</u>: V actually isn't a vector space, b/c a different axiom fails. Can you find an axioms that fails?

Consider ⑥, scalar distributivity.

$$\alpha(u+v) = \alpha \begin{pmatrix} u_1 + v_1 + 1 \\ u_2 + v_2 + 1 \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \alpha v_1 + \alpha \\ \alpha u_2 + \alpha v_2 + \alpha \end{pmatrix}$$

$$\alpha u + \alpha v = \begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix} + \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix} = \begin{pmatrix} \alpha u_1 + \alpha v_1 + 1 \\ \alpha u_2 + \alpha v_2 + 1 \end{pmatrix}$$

choose $\alpha = 2$... can see <u>not</u> the same!

$\therefore$ ⑥ fails $\Rightarrow$ <u>Not</u> a vector space.

3. Determine which of the following are subspaces of P_2 (where P_2 is the vector space of polynomials of degree ≤ 2. e.g., $\{ax^2+bx+c \mid a,b,c \in \mathbb{R}\}$.

<u>Recall</u>: • A subset W of a vector space V is called a <u>subspace</u> of V if W is itself a vector space under the addition & multiplication defined on V.

• <u>Subspace Criterion</u>: A subset $W \subseteq V$ is a subspace of $V \iff$ the following hold:

① W is non-empty.
② W is closed under addition (i.e., $u, v \in W \Rightarrow u+v \in W$).
③ W is closed under scalar multiplication (i.e., $u \in W \Rightarrow \alpha u \in W$ $\forall$ scalars α).

ⓐ $W = \{r(1+x^2) \mid r \in \mathbb{R}\}$.

<u>Non-empty</u>: ① Well, of course we can see W is not empty since $2(1+x^2) \in W$. ✓

<u>"+" closed</u>: ② Suppose $w_1, w_2 \in W$. So, $w_1 = r_1(1+x^2)$ & $w_2 = r_2(1+x^2)$ for some $r_1, r_2 \in \mathbb{R}$. W.T.S. $w_1 + w_2 \in W$.

$$w_1 + w_2 = r_1(1+x^2) + r_2(1+x^2) = \underbrace{(r_1+r_2)}_{\in \mathbb{R}}(1+x^2) \in W. \checkmark$$

<u>"α" closed</u>: ③ Let $\tilde{w} \in W$. So $\tilde{w} = r(1+x^2)$ for some $r \in \mathbb{R}$. W.T.S. $\alpha\tilde{w} \in W$ $\forall \alpha \in \mathbb{R}$.

$$\alpha\tilde{w} = \alpha(r(1+x^2)) = \underbrace{\alpha r}_{\in \mathbb{R}}(1+x^2) \in W.$$

$\therefore$ W is a subspace of V since the 3 conditions hold.

(btw: our vector space is over $\mathbb{R}$, it is usually convention to define the zero locus of 0 to be $\mathbb{R}$.)

in this example, let us use the convention that the roots of the 0 polynomial $(0x^2+0x+0)$ are all real numbers (the x-axis) and not any complex numbers.

(b) $Y = \{$ quadratic polynomials with only real roots $\}$.

Non-empty: (1) x^2-1 has roots $x = \pm 1 \in \mathbb{R} \Rightarrow (x^2-1) \in Y$
$\Rightarrow Y$ is non-empty. ✓

closed under addition: (2) Consider $y_1 = 2x^2$ & $y_2 = -x^2+1$. y_1 has roots $x=0$ & is quadratic $\Rightarrow y_1 \in Y$. y_2 is also quadratic & has roots $x = \pm 1 \Rightarrow y_2 \in Y$. But,
$y_1 + y_2 = 2x^2 + (-x^2+1) = x^2+1$. If we solve for x^2+1's roots we find: $x^2+1=0$
$x^2 = -1 \Rightarrow x = \pm i \notin \mathbb{R}$. ✗

So, $y_1, y_2 \in Y$, but $(y_1+y_2) \notin Y \Rightarrow Y$ not closed under addition $\Rightarrow Y$ is not a subspace of P_2.

Note: Condition (3) holds (can you see why?), but since (2) fails, we know that Y is not a subspace of P_2.

(c) $Z = \{a+bx \mid a,b \in \mathbb{R},\ a^2=b^2\}$.

"+" closed: (2) Consider $z_1 = 2+2x$, $z_2 = -2+2x$. $2^2=2^2$ & $(-2)^2=2^2$, so, $z_1, z_2 \in Z$.
However, $z_1+z_2 = (2+2x)+(-2+2x) = 0+4x$, & $0^2 \neq 4^2 \Rightarrow (z_1+z_2) \notin Z \Rightarrow Z$ is not a subspace of V.

Note: You may check as an exercise that (1) & (3) hold.

(d) $J = \{p+qx+rx^2 \mid p,q,r \in \mathbb{R},\ r \geq 0\}$.

(3) "α" closed

Consider $j = 1+x+2x^2$. $1,1,2 \in \mathbb{R}$ & $2 \geq 0 \Rightarrow j \in J$.
But $-1 * j = -1-x-2x^2$ & $-2 < 0 \Rightarrow -1 * j \notin J \Rightarrow J$ is

not closed under scalar multiplication $\Rightarrow$ J is not a subspace of P_2.

Note: You may check as an exercise that conditions ① & ② hold.

4. Is the set $W_1 = \{(v_1, v_2, 0) \mid v_1, v_2 \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$?

W_1 is a subset of $\mathbb{R}^3$, & is nonempty, since $(1, 2, 0) \in W_1$.

Let $\widetilde{w}_1, \widetilde{w}_2 \in W_1$. So, $\widetilde{w}_1 = (v_1, v_2, 0)$ & $\widetilde{w}_2 = (u_1, u_2, 0)$ for some $v_1, v_2, u_1, u_2 \in \mathbb{R}$.

Then $\widetilde{w}_1 + \widetilde{w}_2 = (v_1, v_2, 0) + (u_1, u_2, 0)$
$= (v_1 + u_1, v_2 + u_2, 0) \in W$

$\Rightarrow$ closed under addition.

Now, W.T.S. W_1 closed under scalar multiplication.
If $\widetilde{w}_1 \in W$, then $\alpha \widetilde{w}_1 = \alpha(v_1, v_2, 0) = (\alpha v_1, \alpha v_2, 0) \in W_1$

$\therefore$ W_1 is a subspace of $\mathbb{R}^3$ since $W_1 \subseteq \mathbb{R}^3$ & our 3 conditions hold.

5. Consider the following sets of vectors:

$$S_1 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} \right\}, \quad S_2 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\},$$

$$S_3 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

ⓐ Which sets span $\mathbb{R}^3$?

<u>Recall</u>: The <u>span</u> of a set $S = \{w_1, \dots, w_r\}$ is the subspace formed by taking all possible linear combinations of the vectors in S.
i.e., $\text{span}(S) = \{\alpha_1 w_1 + \dots + \alpha_r w_r \mid \alpha_1, \dots, \alpha_r \in \mathbb{R}\}$.

(Note: You'll soon see that you always need at least 3 vectors to span $\mathbb{R}^3$... 2 can never do the job).

ⓘ $S_1 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} \right\}$. So, we want to know if S_1 spans $\mathbb{R}^3$; i.e., we want to know if for every vector $v = (a, b, c) \in \mathbb{R}^3$ do there exist scalars $\alpha_1, \alpha_2 \in \mathbb{R}$ s.t. $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}$?

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 9\alpha_1 + \alpha_2 \\ -4\alpha_1 + 3\alpha_2 \\ 2\alpha_1 + 8\alpha_2 \end{pmatrix} = \begin{pmatrix} 9 & 1 \\ -4 & 3 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

So, we want to know:

Is $\begin{pmatrix} 9 & 1 \\ -4 & 3 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ $\underbrace{\text{consistent?}}$

Recall, a linear system is <u>consistent</u> if it has at least one solution.

$$\left[\begin{array}{cc:c} 9 & 1 & a \\ -4 & 3 & b \\ 2 & 8 & c \end{array}\right] \begin{array}{l} r_1 \leftarrow r_1 - \frac{9}{2} r_3 \\ r_2 \leftarrow r_2 + 2 r_3 \end{array}$$

$$\left[\begin{array}{cc:c} 0 & -35 & a - \frac{9}{2}c \\ 0 & 19 & b + 2c \\ 2 & 8 & c \end{array}\right] r_2 \leftarrow r_2 + \tfrac{19}{35} r_1 \qquad \left[\begin{array}{cc:c} 0 & -35 & a \\ 0 & 0 & b + 2c + \frac{19}{35}a - \frac{9}{2} \cdot \frac{19}{35} c \\ 2 & 8 & c \end{array}\right]$$

we could choose values for a, b, c to make this nonzero

$\Rightarrow$ not consistent. e.g., $a = b = c = 1$.

So, if $(a,b,c) = (1,1,1)$, then we have:

$$\begin{bmatrix} 0 & -35 & : & 1 \\ 0 & 0 & : & 11/10 \\ 2 & 8 & : & 1 \end{bmatrix} \swarrow \text{no solution!}$$

So, there do not exist a $\alpha_1, \alpha_2 \in \mathbb{R}$ s.t.

$$\alpha_1 \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ S_1 does not span $\mathbb{R}^3$.

(ii) $S_2 := \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}$.

So, we want to know if S_2 spans $\mathbb{R}^3$; i.e., want to know if the system

$$\begin{bmatrix} 9 & 4 & 0 \\ -4 & 6 & 2 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ is consistent.}$$

Recall: If A is square, then $Ax = b$ is consistent for every $n \times 1$ matrix $b \Leftrightarrow \det(A) \neq 0$.

$$\det\left(\begin{bmatrix} 9 & 4 & 0 \\ -4 & 6 & 2 \\ 2 & -3 & -1 \end{bmatrix}\right) \underset{r_2 \leftarrow r_2 + 2r_3}{=} \det\left(\begin{bmatrix} 9 & 4 & 0 \\ 0 & 0 & 0 \\ 2 & -3 & -1 \end{bmatrix}\right) = 0$$

$\Rightarrow$ our system is not consistent $\Rightarrow$ S_2 does not span $\mathbb{R}^3$.

(iii) $S_3 := \left\{ \begin{pmatrix}9\\-4\\2\end{pmatrix}, \begin{pmatrix}1\\3\\8\end{pmatrix}, \begin{pmatrix}2\\1\\-1\end{pmatrix} \right\}$.

$\begin{pmatrix}a\\b\\c\end{pmatrix}$

$\det\left(\begin{bmatrix}9 & 1 & 2\\-4 & 3 & 1\\2 & 8 & -1\end{bmatrix}\right) = -177 \Rightarrow \begin{pmatrix}9\\-4\\2\end{pmatrix}\alpha_1 + \begin{pmatrix}1\\3\\8\end{pmatrix}\alpha_2 + \begin{pmatrix}2\\1\\-1\end{pmatrix}\alpha_3 \,''$

has a solution for each $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3 \Rightarrow S_3$ spans $\mathbb{R}^3$.

(b) Is the vector $\begin{pmatrix}3\\-1\\2\end{pmatrix}$ in the span of S_1? S_2? S_3?

$S_1 = \left\{ \begin{pmatrix}9\\-4\\2\end{pmatrix}, \begin{pmatrix}1\\3\\8\end{pmatrix} \right\}$. So, $\begin{pmatrix}3\\-1\\2\end{pmatrix} \in \text{span}(S_1)$ if $\exists$

$\alpha_1, \alpha_2 \in \mathbb{R}$ s.t. $\alpha_1 \begin{pmatrix}9\\-4\\2\end{pmatrix} + \alpha_2 \begin{pmatrix}1\\3\\8\end{pmatrix} = \begin{pmatrix}3\\-1\\2\end{pmatrix}$.

So, we want to solve the system: $\begin{bmatrix}9 & 1\\-4 & 3\\2 & 8\end{bmatrix}\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix} = \begin{bmatrix}3\\-1\\2\end{bmatrix}$.

$\left[\begin{array}{cc:c}9 & 1 & 3\\-4 & 3 & -1\\2 & 8 & 2\end{array}\right] \begin{array}{l} r_2 \leftarrow r_2 - 3r_1 \\ r_3 \leftarrow r_3 - 8r_1 \end{array} \left[\begin{array}{cc:c}9 & 1 & 3\\-31 & 0 & -10\\-70 & 0 & -22\end{array}\right] r_2 \leftarrow r_2 - \frac{31}{70} r_3$

$\left[\begin{array}{cc:c}9 & 1 & 3\\0 & 0 & -\frac{9}{35}\\-70 & 0 & -22\end{array}\right] \leftarrow$ no solution!

So, $\begin{pmatrix}3\\-1\\2\end{pmatrix}$ is not is span(S_1).

$$S_2 = \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \right\}.$$

$$\left[\begin{array}{ccc|c} 9 & 4 & 0 & 3 \\ -4 & 6 & 2 & -1 \\ 2 & -3 & -1 & 2 \end{array}\right] \quad r_2 \leftarrow r_2 + 2r_3$$

$$\left[\begin{array}{ccc|c} 9 & 4 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 2 & -3 & -1 & 2 \end{array}\right] \leftarrow \text{no solution!}$$

So $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \notin \text{span}(S_2)$.

$$S_3 = \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

$$\left[\begin{array}{ccc|c} 9 & 1 & 2 & 3 \\ -4 & 3 & 1 & -1 \\ 2 & 8 & -1 & 2 \end{array}\right] \quad \begin{array}{l} r_1 \leftarrow r_1 + 2r_3 \\ r_2 \leftarrow r_2 + r_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 13 & 17 & 0 & 7 \\ -2 & 11 & 0 & 1 \\ 2 & 8 & -1 & 2 \end{array}\right] \quad r_3 \leftarrow r_3 + r_2$$

$$\left[\begin{array}{ccc|c} 13 & 17 & 0 & 7 \\ -2 & 11 & 0 & 1 \\ 0 & 19 & -1 & 3 \end{array}\right]$$

$$-z = 3 - 19y \Rightarrow z = 19y - 3 \Rightarrow z = 19\left(\frac{9}{59}\right) - 3 = -\frac{6}{59}$$

$$13x + 17y = 7 \Rightarrow 13x = -17y + 7 \Rightarrow x = -\frac{17}{13}y + \frac{7}{13}$$

$$-2x + 11y = 1 \Rightarrow -2\left(-\frac{17}{13}y + \frac{7}{13}\right) + 11y = 1$$

$$\Rightarrow \frac{34}{13}y - \frac{14}{13} + \frac{143}{13}y = 1 \Rightarrow \frac{177}{13}y = \frac{27}{13} \Rightarrow y = \frac{27}{177} = \frac{9}{59}$$

$$\Rightarrow x = -\frac{17}{13}\left(\frac{9}{59}\right) + \frac{7}{13} = \frac{20}{59}.$$

So, $\frac{20}{59}\begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix} + \frac{9}{59}\begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} - \frac{6}{59}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \in \text{span}(S_3)$.

(c) Which of these sets are linearly independent?

<u>Recall</u>: If a set of vectors $S = \{v_1, \dots, v_r\}$ is such that the equation $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r = \bar{0}$ has only the trivial solution ($\underline{\text{i.e.}}$, $\alpha_1 = \dots = \alpha_r = 0$), then these vectors are said to be <u>linearly independent</u>. If there exist nontrivial solutions, then the vectors are said to be <u>linearly dependent</u>.

<u>S_1:</u> So, for $S_1 = \left\{ \begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} \right\}$ we want to know if there are nontrivial solutions to the equation

$$\begin{pmatrix} 9 \\ -4 \\ 2 \end{pmatrix} \alpha_1 + \begin{pmatrix} 1 \\ 3 \\ 8 \end{pmatrix} \alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{array}{cc|c} 9 & 1 & 0 \\ -4 & 3 & 0 \\ 2 & 8 & 0 \end{array}\right] \begin{array}{l} r_2 \leftarrow r_2 - 3r_1 \\ r_3 \leftarrow r_3 - 8r_1 \end{array} \left[\begin{array}{cc|c} 9 & 1 & 0 \\ -31 & 0 & 0 \\ -70 & 0 & 0 \end{array}\right] \begin{array}{l} r_2 \leftarrow r_2 \cdot -\frac{1}{31} \\ r_3 \leftarrow r_3 \cdot -\frac{1}{70} \end{array}$$

$$\left[\begin{array}{cc|c} 9 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \begin{array}{l} r_3 \leftarrow r_3 - r_2 \\ r_1 \leftarrow r_1 - 9r_2 \end{array} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} \alpha_1 = 0 \\ \alpha_2 = 0 \end{array}$$

(<u>Note</u>: If you just have 2 vectors in $\mathbb{R}^n$ with the usual addition rule, then they're dependent $\Leftrightarrow$ they differ by a scalar multiple).

So, this system has only the trivial solution $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ S_1 is a linearly independent set.

<u>S_2:</u> Now, let's check to see if S_2 is a lin. ind. set.

$\underline{\text{i.e.}}$, does $\begin{bmatrix} 9 & 4 & 0 \\ -4 & 6 & 2 \\ 2 & -3 & -1 \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ have nontrivial solutions?

Recall that $A\vec{x}=0$ has only the trivial solution $\Leftrightarrow$ A is invertible (if A is a $n \times n$ matrix).

$$\det\left(\begin{bmatrix} 9 & 4 & 0 \\ -4 & 6 & 2 \\ 2 & -3 & -1 \end{bmatrix}\right) = 0 \Rightarrow A \text{ \underline{not} invertible} \Rightarrow$$

$\exists$ nontrivial solutions $\Rightarrow S_2$ is not a linearly independent set (i.e., S_2 is a linearly dependent set).

$\underline{S_3}$:

$$\det\left(\begin{bmatrix} 9 & 1 & 2 \\ -4 & 3 & 1 \\ 2 & 8 & -1 \end{bmatrix}\right) = -177 \Rightarrow A \text{ invertible}$$

$\Rightarrow A\vec{x}=0$ has only the trivial solution

$\Rightarrow S_3$ is a linearly independent set.